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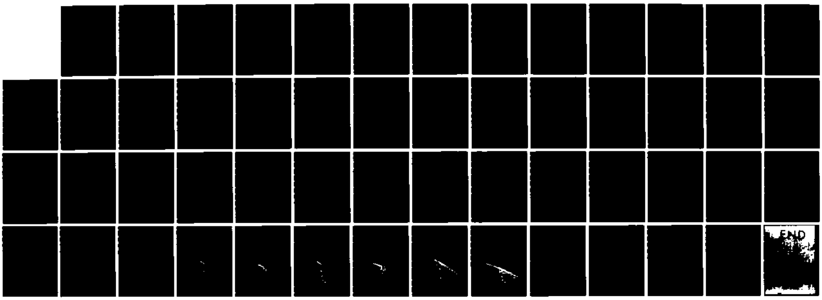

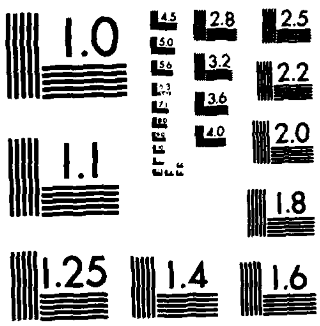

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ANALYTIC SOLUTION OF THE SPENCER-LEWIS
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An exact solution for the angular-spatial moments of the Spencer-Lewis equation is given, along with methods for reconstructing the electron density function. The new solution technique is implemented in a computer code and several sample calculations are carried out with monoenergetic plane sources of electrons in infinite media of aluminum and carbon. The sample calculations produce the electron distribution in space and path length rather than some integrated quantity such as the energy deposition profile. Such analytic solutions are intended to serve as benchmarks for testing numerical electron transport codes.

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ANALYTIC SOLUTION OF THE SPENCER-LEWIS

ANGULAR-SPATIAL MOMENTS EQUATIONS

I. INTRODUCTION

The major accomplishments of the project were

1. A closed form solution for the angular-spatial moments of the Spencer-Lewis equation[1,2]

2. A computer program that generates benchmark solutions for electron densities as a function of position and path length due to monoenergenic plane sources in infinite media.

The original aim of the proposed work was to use benchmark calculations to aid in the development of numerical codes for solving electron transport problems. It was proposed to determine energy deposition profiles using Spencer's method of moments[2]. Such solutions would then be used to evaluate some of the numerical approximations used in the streaming ray[3-6] (SR) and SN[6-8] methods. Spencer's technique consists of reconstructing energy deposition profiles from angular-spatial-path length moments.

During the course of this project it was discovered that the angular-spacial moments could be determined directly, thereby eliminating the need to take moments in path length. At this point it was decided to change the emphasis of the work. Instead of concentrating on the application of existing benchmark methods, the primary objective become the development

of new benchmark solutions based on the new closed form solution for the angular-spatial moments.

The major effort was devoted to calculating the angular-spatial moments. The remaining work involved reconstructing electron density distributions from the moments. Several sample calculations have been carried out with monoenergetic plane sources of electrons in infinite media of aluminum and carbon.

An important feature of the new solutions is that they provide the electron density distribution in position and path length rather than an integrated quantity such as the energy deposition profile. With a little more programming it appears that the angular electron density could be determined as well.

II. THEORY

This section begins with an outline of Spencer's method of moments. It is then shown that the set of coupled ordinary differential equations for the angular-spatial moments can be solved directly, thereby eliminating the need for path length moments. Finally some of the standard techniques for reconstructing the electron density from its moments are discussed.

II. A. Spencer's Method of Moments

The electron distribution in space, angle and path length can be obtained by solving the Spencer-Lewis equation. In one

dimension this equation is[1,2]

$$\left(\frac{\partial}{\partial s} + \mu \frac{\partial}{\partial x} + \sigma\right) \phi(x,s,\mu)$$

$$= 2\pi \int_{-1}^{+1} \sigma(\mu'\to\mu)\, \phi(x,s,\mu')\, d\mu' + Q(x,s,\mu), \quad (1)$$

where

$\phi(x,s,\mu)$ = the density of electrons as a function of position x, path length s, and direction μ,

σ = total interaction cross section,

$\sigma(\mu'\to\mu)$ = differential scattering cross section,

$Q(x,s,\mu)$ = the fixed source of electrons.

We will assume that the medium is infinite and homogeneous, that initially there are no electrons present,

$$\phi(x,0^-,\mu) = 0 \quad , \quad (2)$$

and that Q is a function of the form

$$Q(x,s,\mu) = \delta(x)\, \delta(s)\, Q(\mu) \quad . \quad (3)$$

It is convenient to measure x and s in terms of the maximum electron range. Then the region of interest is defined by $-1 \leq x \leq 1$ and $0 \leq s \leq 1$.

To solve Eq. (1), let

$$\phi(x,s,\mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, \phi_\ell(x,s)\, P_\ell(\mu), \quad (4)$$

A-1

$$\sigma(s,\mu'\to\mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, \sigma_\ell(s) P_\ell(\mu_0) \ , \tag{5}$$

and

$$Q(x,s,\mu) = \delta(x)\ \delta(s) \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, q_\ell\ P_\ell(\mu) \ , \tag{6}$$

where μ_0 is the cosine of the angle between directions μ and μ', and the P_ℓ are the Legendre polynomials.

Using the addition theorem and recurrence relation for Legendre polynomials, and Eqs. (1) and (3) to (6), one obtains

$$\left(\frac{\partial}{\partial s} + \sigma - \sigma_\ell\right)\ \phi_\ell(x,s) + \frac{\ell}{2\ell+1}\,\frac{\partial}{\partial x}\,\phi_{\ell-1}(x,s)$$

$$+\ \frac{\ell+1}{2\ell+1}\,\frac{\partial}{\partial x}\,\phi_{\ell+1}(x,s) = \delta(x)\ \delta(s)\ q_\ell,\ \ell = 0, 1, 2, \ldots \tag{7}$$

Since the electrons originate at x = 0, it follows that

$$\phi(x,s,\mu) = 0,\ |x| > s\ . \tag{8}$$

Thus multiplying Eq. (7) by x^n and integrating over the interval $-1 < x < 1$, yields the result

$$\left[\frac{d}{ds} + \sigma(s) - \sigma_\ell(s)\right]\phi_{\ell n}(s) - \frac{n}{2\ell+1}\Big[\ell\ \phi_{\ell-1,n-1}(s) +$$

$$(\ell+1)\ \phi_{\ell+1,n-1}(s)\Big] = \delta_{n,0}\ \delta(s)\ q_\ell,\ n,\ell = 0, 1, 2, \ldots\ , \tag{9}$$

where

$$\phi_{\ell n}(s) = \int_{-1}^{1} x^n\ \phi_\ell(x,s)\ dx\ . \tag{10}$$

The s dependence of the cross sections appearing in Eq. (9) is reasonably well modeled[2] by the relation,

$$\sigma(s) - \sigma_\ell(s) \approx \frac{f_\ell}{1-s} , \tag{11}$$

where

$$f_\ell = \sigma(0) - \sigma_\ell(0) . \tag{12}$$

Using Eq. (11), multiplying Eq. (8) by $(1-s)^{p+1}$, and integrating from 0^- to 1 yields Spencer's result,

$$\phi^p_{\ell n} = \frac{\delta_{no} q_o}{p+1+f_\ell} + \frac{n\left[\ell\phi^{p+1}_{\ell-1,n-1} + (\ell+1)\phi^{p+1}_{\ell+1,n-1}\right]}{(p+1+f_\ell)(2\ell+1)} , p > -1 , \tag{13}$$

where

$$\phi^p_{\ell n} = \int_{0^-}^{1} (1-s)^p \, \phi_{\ell n}(s) ds . \tag{14}$$

Surface terms do not appear in Eq. (13) because the factor $(1-s)^{(p+1)}$, $p > -1$ vanishes at the upper limit, and as shown in Section II. B., $\phi_{\ell n}(s)$ vanishes at the lower limit. The $\phi^p_{\ell n}$, $p > -1$ are easily generated from Eq. (13).

II. B. Closed Form Expression for the Angular-Spatial- Moments

The simple recursion relationship for the $\phi^p_{\ell n}$, Eq. (13) was made possible by assuming that the cross sections vary inversely with the residual electron range, Eq. (11). It is shown here that this same assumption enables one to solve Eqs. (9) directly.

For the case n=0 we have

$$\left(\frac{d}{ds} + \frac{f_\ell}{1-s}\right) \phi_{\ell 0}(s) = q_\ell \, \delta(s) \quad . \tag{15}$$

From Eqs. (2), (4) and (10) it follows that

$$\phi_{\ell n}(0^-) = 0 \ , \quad n=0, 1, 2, \ldots \quad . \tag{16}$$

Thus, the solution of Eq. (15) is easily shown to be

$$\phi_{\ell 0}(s) = \begin{cases} q_\ell \; \theta(s) \; (1-s)^{f_\ell} \ , & \ell>0 \ , \\ q_0 \; \theta(s) \ , & \ell=0 \ , \end{cases} \tag{17}$$

where $\theta(s)$ is the step function.

For $n > 0$,

$$\left(\frac{d}{ds} + \frac{f_\ell}{1-s}\right) \phi_{\ell n}(s) = \frac{n}{2\ell+1} \left[\ell \phi_{\ell-1,n-1}(s) + (\ell+1) \; \phi_{\ell+1,n-1}(s)\right] . \tag{18}$$

To solve this equation it is necessary to know the initial values of the $\phi_{\ell n}(s)$ as s approaches zero from the right rather than from the left.

Using Eq. (10),

$$\lim_{s\to 0^+} \phi_{\ell n}(s) = \lim_{s\to 0^+} \int_{-1}^{1} x^n \; \phi_\ell(x,s)dx \ . \tag{19}$$

From Eqs. (4) and (8)

$$\phi_\ell(x,s) = 0 \ , \quad |x| > s \ , \tag{20}$$

therefore

$$\phi_{\ell n}(0^+) = 0, \quad n > 0 \ . \tag{21}$$

In spite of Eq. (20), $\phi_{\ell 0}(0^+) \neq 0$, as seen by Eq. (17). Thus, $\phi_\ell(x,0)$ must contain a delta function centered at $x = 0$ of strength q_ℓ. This should not be surprising in light of Eq. (6).

After multiplying by $(1-s)^{-f_\ell}$ Eq. (18) can be easily integrated to yield

$$\phi_{\ell n}(s) = n(1-s)^{f_\ell}\int_0^s \frac{\ell\phi_{\ell-1,n-1}(t) + (\ell+1)\phi_{\ell+1,n-1}(t)}{(2\ell+1)(1-t)^{f_\ell}}\,dt, \qquad n > 0 \tag{22}$$

where use has been made of Eq. (21).

Consider now the anzatz:

$$\phi_{\ell n}(s) = \sum_{i=I_{\ell n}}^{n} \sum_{j=0}^{J_{ni}} A_{\ell nij}\,(1-s)^{(f_\ell+i+|i|+2j)}\ , \tag{23}$$

where

$$I_{\ell n} = \max\{-n,-\ell\}\ , \tag{24}$$

$$J_{ni} = \begin{cases} (n-|i|)/2 & ,\ n-|i| \text{ even}\ , \\ (n-|i|-1)/2 & ,\ n-|i| \text{ odd}\ , \end{cases} \tag{25}$$

and the $A_{\ell nij}$ are constants to be determined.

Eqs. (17) and (23) are consistent provided

$$A_{\ell 000} = q_\ell. \tag{26}$$

After some rather tedious algebra it can be shown that Eqs. (22) and (23) are equivalent provided the $A_{\ell nij}$, $n > 0$ are given recursively by

$$A_{\ell nij} = \begin{cases} C_{\ell nij} + E_{\ell nij} , & I_{\ell,n-2} \leq i \leq n-2 \ , \ j \neq 0 \ , \\ C_{\ell nij} \ , & I_{\ell n} \leq i < I_{\ell,n-2} \ , \ j \neq 0 \ , \\ E_{\ell nij} \ , & 1 \leq i \leq n \ , \ j=0 \ , \\ C_{\ell nij} \ , & I_{\ell n} \leq i < 0 \ , \ j=0 \ , \\ -\sum\limits_{\substack{i'=I_{\ell n} \\ i \neq 0}}^{n} \sum\limits_{j'=0}^{J_{ni'}} A_{\ell n i' j'} \ , & i=j=0 \ , \end{cases} \tag{27}$$

where

$$C_{\ell nij} = \begin{cases} -\dfrac{n\ell A_{\ell-1,n-1,i+1,j-1}}{(2\ell+1)(f_{\ell+i}+|i|+2j-f_{\ell})} \ , & 0 \leq i \leq n-2 \ , \ j \neq 0 \ , \\ -\dfrac{n\ell A_{\ell-1,n-1,i+1,j}}{(2\ell+i)(f_{\ell+i}+|i|+2j-f_{\ell})} \ , & I_{\ell n} \leq i < 0 \ , \end{cases} \tag{28}$$

and

$$E_{\ell nij} = \begin{cases} -\dfrac{n(\ell+1)A_{\ell+1,n-1,i-1,j}}{(2\ell+1)(f_{\ell+i}+|i|+2j-f_{\ell})} \ , & 1 \leq i \leq n \ , \\ -\dfrac{n(\ell+1)A_{\ell+1,n-1,i-1,j-1}}{(2\ell+1)(f_{\ell+i}+|i|+2j-f_{\ell})} \ , & I_{\ell,n-2}, \leq i \leq 0 \ , \ j \neq 0 \ , \end{cases} \tag{29}$$

II. C. Computational Considerations

Equations (23) to (29) constitute an exact solution for the $\phi_{\ell n}(s)$. We now discuss some of the details of their numerical implementation.

Calculation of the $A_{\ell nij}$:

From Eqs. (23) to (25) it is evident that the $A_{\ell nij}$ comprise a very sparse array. On the other hand, as seen from Eqs. (27)

to (29) the $A_{\ell nij}$ depend only on the $A_{\ell,n-1,ij}$. Thus to avoid excessive storage requirements, the $A_{\ell,n-1,ij}$ are discarded after the $A_{\ell nij}$ are determined. In addition, for a given n value the $A_{\ell nij}$ are stored in a one-dimensional full array whose index, $k(\ell,n,i,j)$ is given by

$$k = \sum_{\ell'=0}^{\ell-1} \sum_{i'=I_{\ell' n}}^{n} (J_{ni'}+1)$$

$$+ \sum_{i'=I_{\ell n}}^{i-1} (J_{ni'}+1) + j + 1 \tag{30}$$

It is possible to evaluate the summations analytically, however the algebra is quite lengthy and will not be reproduced here. The final expression for k can be presented most compactly if the rules for integer arithmetic (any quotient is truncated to an integer value) are assumed to apply. In this case

$$k = k_1 + k_2 + j + 1 \tag{31}$$

where

$$k_1 = \begin{cases} 0, & \ell = 0, \\ \ell + \dfrac{n[n(12\ell-2n-9)+36\ell-10]}{24}, & \ell > n, \\ 1 + \dfrac{6n\ell(n+\ell+3)+(\ell-1)\,[(\ell-1)(1-2\ell)+26]}{24}, & 0 < \ell \le n, \end{cases} \tag{32}$$

$$k_2 = \begin{cases} 0, & i = I_{\ell n}, \\ 1 + \dfrac{(2[(i-I_{\ell n})\cdot(n+1)-1]-(i-1)^2-I_{\ell n}^2)}{4}, & I_{\ell n} < i \leq 1, \\ \dfrac{2[(i-1_{\ell n})\cdot(n+1)+i]+(i-1)^2-I^2_{\ell n}}{4}, & i > 1 . \end{cases} \quad (33)$$

III. RECONSTRUCTION

The electron density distribution can be obtained via Eq. (4) if the $\phi_\ell(x,s)$ are known. At least on physical grounds, one would expect the $\phi_\ell(x,s)$ as functions x to belong to $L^2(-1,1)$ space for any s value. Therefore, in theory it should be possible to reconstruct (with convergence in the mean) $\phi_\ell(x,s)$ from its moments $\phi_{\ell n}(s)$, n = 0, 1, 2, ... Since the $\phi_{\ell n}(s)$ are determined recursively, in practice one must approximate $\phi_\ell(x,s)$ from a finite number, N of its moments. We consider three straight forward techniques.

1. In the first method (and also in the second) we exploit the fact that there is a wave front located at $|x|=s$ [see Eq. (20)] and put

$$\phi_\ell(x,s) = \begin{cases} \displaystyle\sum_{m=0}^{N} \frac{2m+1}{2} H_{\ell m}(s)\, P_m\left(\frac{x}{s}\right), & |x| \leq s \\ 0, & |x| > s, \end{cases} \quad (34)$$

where

$$H_{\ell m}(s) = \frac{1}{s}\int_{-s}^{s} P_m(\tfrac{x}{s})\ \phi_\ell(x,s)dx \ , \tag{35}$$

and the Legendre polynomials, $P_m(\frac{x}{s})$ form an orthogonal basis on $[-s,s]$. If the coefficients, h_{nm} are defined such that

$$P_m(x) = \sum_{n=0}^{m} h_{nm}\ x^n \quad , \tag{36}$$

then

$$H_{\ell m}(s) = \sum_{n=0}^{m} \frac{h_{nm}}{s^{n+1}} \int_{-s}^{s} x^n \phi_\ell(x,s)dx \ . \tag{37}$$

From Eqs. (10) and (20)

$$H_{\ell m}(s) = \sum_{n=0}^{m} \frac{h_{nm}}{s^{n+1}}\ \phi_{\ell n}(s) \quad . \tag{38}$$

2. The second procedure to be considered is mathematically equivalent to first provided all calculations are carried out with infinite precision. Therefore the numerical discrepancies between the two techniques should give some indication as to the severity of computer round off errors.

In place of Eq. (34) we set

$$\phi_\ell(x,s) = \begin{cases} \displaystyle\sum_{m=0}^{N} G_{\ell m}\ x^m \ , & |x| \le s \ , \\ 0 \ , & |x| \ge s \ , \end{cases} \tag{39}$$

Multiplying Eq. (35) by x^n and integrating from $-s$ to s gives, in

matrix form

$$\vec{\phi}_\ell = X \vec{G}_\ell \tag{40}$$

where $\vec{G}_\ell$ and $\vec{\phi}_\ell$ are N-dimensional column vectors whose m th components are $G_{\ell m}$ and $\phi_{\ell m}(s)$ respectively, and X is an N x N matrix whose n,m th element is given by

$$x_{nm} = \int_{-s}^{s} x^{n+m}\, dx = \begin{cases} \dfrac{2s^{n+m+1}}{n+m+1}, & n+m \text{ even}, \\ 0, & n+m \text{ odd}. \end{cases} \tag{41}$$

Inverting Eq. (40),

$$\vec{G}_\ell = X^{-1} \vec{\phi}_\ell . \tag{42}$$

3. The final reconstruction method is identical to the second except that the location of the wave front is not explicitly built into the procedure. The equations for this case are the same as those for method 2, except that Eq. (39) becomes

$$\phi_\ell(x,s) = \sum_{m=0}^{N} G_{\ell m}\, x^m \tag{43}$$

and Eq. (41) becomes

$$x_{nm} = \int_{-1}^{1} x^{n+m}\, dx = \begin{cases} \dfrac{2}{n+m+1}, & n+m \text{ even}, \\ 0, & n+m \text{ odd}. \end{cases} \tag{44}$$

While methods 1 and 2 are theoretically identical for all values of N, this last method should agree (convergence in the mean) with the former two only in the limit $N \to \infty$.

IV. NUMERICAL RESULTS

A computer program was written which determines the moments $\phi^p_{\ell n}$, and the total electron density, $\phi(x,s)$ defined by

$$\phi(x,s) = 2\pi\int_{-1}^{1} \phi(x,s,\mu)\, d\mu = \phi_0(x,s), \tag{45}$$

where the second equality follows from Eq. (4). The $\phi^p_{\ell n}$ are determined both by Spencer's method, Eq. (13), and also from Eq. (23). The total density distribution is reconstructed by all three methods described in Section II. C., namely; Eqs. (34), (39), and (43).

IV. A. Cross Sections

The moments method appears to be restricted to infinite media problems, and to cross sections of the form specified in Eq. (11), or the somewhat more complicated expression[2,9]

$$\sigma(s)-\sigma_\ell(s) = \frac{\alpha f_\ell}{(1-s)(1-s-\alpha)} , \tag{46}$$

where α is an adjustable parameter. Due to these limitations, moments calculations are probably more useful today for benchmarking more flexible numerical techniques, such as Monte Carlo[10] or discrete ordinates[5-8], than for producing realistic electron transport results. For this reason our results are based on Eq. (11) rather than the more complicated Eq. (46). In References 2 and 9 Spencer gives the counterpart of Eq. (13) for the case when the cross sections satisfy Eq. (46).

Although Eq. (11) is of limited validity it does permit the cross section for the incident (s=0) electron energy to be specified to any desired accuracy. The model assumed in all calculations is the following expression for screened Rutherford elastic scattering[2],

$$\sigma(s=0,\, \mu_0) = \frac{2\pi Z(Z+1)}{A} N_a\, r_0^2 \left[\frac{E_0+1}{E_0(E_0+2)}\right]^2 \cdot \left[\frac{1}{(1+\eta-\mu_0)^2}\right], \qquad (47)$$

where

E_0 = kinetic energy of source electrons (s=0) in mc^2 units,

Z = atomic number of the transport medium,

A = atomic weight of the transport medium,

N_a = Avogadro's number,

r_0 = e^2/mc^2 (classical electron radius),

μ_0 = cosine of the scattering angle,

and the screening constant, η, is given by the Moliere formula[9]

$$\eta = 0.5 \left[\frac{Z^{1/3}}{.855(137)}\right]^2 \left[\frac{1}{E_0(E_0+2)}\right] \left[1.13 + 3.76\left(\frac{Z}{137}\right)^2 \frac{(E_0+1)^2}{E_0(E_0+2)}\right] \qquad (48)$$

As specified in Eq. (48), the units of the cross section are (cm^2/g). To be consistent with our former development where distances are measured in range units we let

$$\sigma(s=0,\, \mu_0) \rightarrow \sigma(s=0,\, \mu_0) \cdot S(E_0)\ , \qquad (49)$$

where $S(E_0)$ is the maximum range in (gm/cm^2) of electrons of energy E_0.

IV. B. Moments

According to Eq. (23) and the definition of $\phi^{p}_{\ell n}$, Eq. (14)

$$\phi^{p}_{\ell n} = \sum_{i=1_{\ell n}}^{n} \sum_{j=0}^{j_{in}} \frac{A_{\ell nij}}{f_{\ell+i} + |i| + 2j + p + 1} \tag{50}$$

if $f_{\ell+i} + |i| + 2j + p + 1 > 0$ for all ℓ, i, j, p and diverges otherwise. For the case $\ell = i = j = 0$ the denominator inside the double summation becomes p+1 ($f_0 = \sigma - \sigma_0 = 0$) and therefore, Eq. (50) has the restriction $p > -1$, which is identical to the restriction placed on Eq. (13). The $\phi^{p}_{\ell n}$ were calculated for ℓ,p = 0, 1, 2, 3 and n = 0, 1, ... 40 with an isotropic ($q_\ell = \delta_{\ell 0}$) source of electrons and initial energies, E_0 = 1.0 MeV and 0.1 MeV. As shown in the left half of Tables I and II the results obtained from Eqs. (13) and (50) are in agreement to at least five significant figures provided that the calculations for Eq. (50) are carried out in double percision (on a CYBER 175 computer). However, Eq. (50) does appear to be sensitive to round off errors as seen in the second column of results in Tables I and II, which were obtained using single percision.

An outside verification of the new formulation would be desirable. Unfortunately, Spencer's published moments values[10] are based on Eq. (46) rather than Eq. (11). Nevertheless, as $\alpha \to \infty$ the two cross section expressions become identical. Comparison of moments determined from Eq. (50) and Spencer's published results are shown in Table III for an isotropic plane source, ℓ

TABLE I

Angular-spatial-pathlength moments as determined by Eqs. (13) and (50) with a plane isotropic source of 1.0 MeV electrons.

n	From Eq. (50) Double Precision or Eq. (13) $\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	From Eq. (50) Single Precision $\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	p
0	.10000+01	.00000+00	.00000+00	.00000+00	.10000+01	.00000+00	.00000+00	.00000+00	0
	.50000+00	.00000+00	.00000+00	.00000+00	.50000+00	.00000+00	.00000+00	.00000+00	1
	.33333+00	.00000+00	.00000+00	.00000+00	.33333+00	.00000+00	.00000+00	.00000+00	2
	.25000+00	.00000+00	.00000+00	.00000+00	.25000+00	.00000+00	.00000+00	.00000+00	3
1	.00000+00	.34662-01	.00000+00	.00000+00	.00000+00	.34662-01	.00000+00	.00000+00	0
	.00000+00	.19130-01	.00000+00	.00000+00	.00000+00	.19130-01	.00000+00	.00000+00	1
	.00000+00	.12240-01	.00000+00	.00000+00	.00000+00	.12240-01	.00000+00	.00000+00	2
	.00000+00	.85380-02	.00000+00	.00000+00	.00000+00	.85380-02	.00000+00	.00000+00	3
2	.38260-01	.00000+00	.14322-02	.00000+00	.38260-01	.00000+00	.14322-02	.00000+00	0
	.12240-01	.00000+00	.83797-03	.00000+00	.12240-01	.00000+00	.83797-03	.00000+00	1
	.56920-02	.00000+00	.53844-03	.00000+00	.56920-02	.00000+00	.53844-03	.00000+00	2
	.31536-02	.00000+00	.36870-03	.00000+00	.31536-02	.00000+00	.36870-03	.00000+00	3
3	.00000+00	.28942-02	.00000+00	.59632-04	.00000+00	.28942-02	.00000+00	.59632-04	0
	.00000+00	.11654-02	.00000+00	.36307-04	.00000+00	.11654-02	.00000+00	.36307-04	1
	.00000+00	.57151-03	.00000+00	.23622-04	.00000+00	.57151-03	.00000+00	.23622-04	2
	.00000+00	.31645-03	.00000+00	.16141-04	.00000+00	.31645-03	.00000+00	.16141-04	3

Column header spanning all moment columns: $\phi^{p}_{\ell n}$

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
	.46615-02	.00000+00	.18266-03	.00000+00	.46615-02	.00000+00	.18266-03	.00000+00	0
4	.11430-02	.00000+00	.83105-04	.00000+00	.11430-02	.00000+00	.83105-04	.00000+00	1
	.42194-03	.00000+00	.42968-04	.00000+00	.42194-03	.00000+00	.42968-04	.00000+00	2
	.19053-03	.00000+00	.24285-04	.00000+00	.19053-03	.00000+00	.24285-04	.00000+00	3
	.00000+00	.45381-03	.00000+00	.10097-04	.00000+00	.45381-03	.00000+00	.10097-04	0
5	.00000+00	.14573-03	.00000+00	.49792-05	.00000+00	.14573-03	.00000+00	.49792-05	1
	.00000+00	.58532-04	.00000+00	.26912-05	.00000+00	.58532-04	.00000+00	.26912-05	2
	.00000+00	.27099-04	.00000+00	.15577-05	.00000+00	.27099-04	.00000+00	.15577-05	3
	.87440-03	.00000+00	.34410-04	.00000+00	.87440-03	.00000+00	.34410-04	.00000+00	0
6	.17560-03	.00000+00	.12851-04	.00000+00	.17560-03	.00000+00	.12851-04	.00000+00	1
	.54197-04	.00000+00	.55690-05	.00000+00	.54197-04	.00000+00	.55690-05	.00000+00	2
	.20805-04	.00000+00	.26826-05	.00000+00	.20805-04	.00000+00	.26826-05	.00000+00	3
	.00000+00	.97685-04	.00000+00	.21925-05	.00000+00	.97685-04	.00000+00	.21925-05	0
7	.00000+00	.26247-04	.00000+00	.90738-06	.00000+00	.26247-04	.00000+00	.90738-06	1
	.00000+00	.89692-05	.00000+00	.41858-06	.00000+00	.89692-05	.00000+00	.41858-06	2
	.00000+00	.35819-05	.00000+00	.20968-06	.00000+00	.35819-05	.00000+00	.20968-06	3
	.20998-03	.00000+00	.82679-05	.00000+00	.20998-03	.00000+00	.82679-05	.00000+00	0
8	.35877-04	.00000+00	.26281-05	.00000+00	.35877-04	.00000+00	.26281-05	.00000+00	1
	.95516-05	.00000+00	.98290-06	.00000+00	.95516-05	.00000+00	.98290-06	.00000+00	2
	.31994-05	.00000+00	.41337-06	.00000+00	.31994-05	.00000+00	.41337-06	.00000+00	3
	.00000+00	.25664-04	.00000+00	.57676-06	.00000+00	.25664-04	.00000+00	.57676-06	0
9	.00000+00	.59488-05	.00000+00	.20605-06	.00000+00	.59488-05	.00000+00	.20605-06	1
	.00000+00	.17741-05	.00000+00	.83014-07	.00000+00	.17741-05	.00000+00	.83014-07	2
	.00000+00	.62438-06	.00000+00	.36677-07	.00000+00	.62438-06	.00000+00	.36677-07	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
10	.59488-04	.00000+00	.23426-05	.00000+00	.59488-04	.00000+00	.23426-05	.00000+00	0
	.88703-05	.00000+00	.64990-06	.00000+00	.88703-05	.00000+00	.64990-06	.00000+00	1
	.20813-05	.00000+00	.21423-06	.00000+00	.20813-05	.00000+00	.21423-06	.00000+00	2
	.61959-06	.00000+00	.80085-07	.00000+00	.61959-06	.00000+00	.80085-07	.00000+00	3
11	.00000+00	.77555-05	.00000+00	.17434-06	.00000+00	.77555-05	.00000+00	.17434-06	0
	.00000+00	.15843-05	.00000+00	.54898-07	.00000+00	.15843-05	.00000+00	.54898-07	1
	.00000+00	.41994-06	.00000+00	.19661-07	.00000+00	.41994-06	.00000+00	.19661-07	2
	.00000+00	.13233-06	.00000+00	.77792-08	.00000+00	.13233-06	.00000+00	.77792-08	3
12	.19012-04	.00000+00	.74870-06	.00000+00	.19012-04	.00000+00	.74870-06	.00000+00	0
	.25197-05	.00000+00	.18462-06	.00000+00	.25197-05	.00000+00	.18462-06	.00000+00	1
	.52933-06	.00000+00	.54489-07	.00000+00	.52933-06	.00000+00	.54489-07	.00000+00	2
	.14201-06	.00000+00	.18357-07	.00000+00	.14201-06	.00000+00	.18357-07	.00000+00	3
13	.00000+00	.26035-05	.00000+00	.58528-07	.00000+00	.26035-05	.00000+00	.58528-07	0
	.00000+00	.47622-06	.00000+00	.16502-07	.00000+00	.47622-06	.00000+00	.16502-07	1
	.00000+00	.11375-06	.00000+00	.53263-08	.00000+00	.11375-06	.00000+00	.53263-08	2
	.00000+00	.32488-07	.00000+00	.19102-08	.00000+00	.32488-07	.00000+00	.19102-08	3
14	.66671-05	.00000+00	.26255-06	.00000+00	.66671-05	.00000+00	.26255-06	.00000+00	0
	.79626-06	.00000+00	.58342-07	.00000+00	.79626-06	.00000+00	.58342-07	.00000+00	1
	.15161-06	.00000+00	.15607-07	.00000+00	.15161-06	.00000+00	.15607-07	.00000+00	2
	.37054-07	.00000+00	.47900-08	.00000+00	.37054-07	.00000+00	.47900-08	.00000+00	3
15	.00000+00	.94935-06	.00000+00	.21342-07	.00000+00	.94935-06	.00000+00	.21342-07	0
	.00000+00	.15739-06	.00000+00	.54540-08	.00000+00	.15739-06	.00000+00	.54540-08	1
	.00000+00	.34248-07	.00000+00	.16037-08	.00000+00	.34248-07	.00000+00	.16037-08	2
	.00000+00	.89520-08	.00000+00	.52636-09	.00000+00	.89520-08	.00000+00	.52636-09	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
16	.25182-05	.00000+00	.99166-07	.00000+00	.25182-05	.00000+00	.99166-07	.00000+00	0
	.27398-06	.00000+00	.20075-07	.00000+00	.27398-06	.00000+00	.20075-07	.00000+00	1
	.47744-07	.00000+00	.49148-08	.00000+00	.47744-07	.00000+00	.49148-08	.00000+00	2
	.10723-07	.00000+00	.13862-08	.00000+00	.10723-07	.00000+00	.13862-08	.00000+00	3
17	.00000+00	.37021-06	.00000+00	.83225-08	.00000+00	.37021-06	.00000+00	.83225-08	0
	.00000+00	.56170-07	.00000+00	.19465-08	.00000+00	.56170-07	.00000+00	.19465-08	1
	.00000+00	.11233-07	.00000+00	.52599-09	.00000+00	.11233-07	.00000+00	.52599-09	2
	.00000+00	.27085-08	.00000+00	.15926-09	.00000+00	.27085-08	.00000+00	.15925-09	3
18	.10111-05	.00000+00	.39816-07	.00000+00	.10111-05	.00000+00	.39816-07	.00000+00	0
	.10110-06	.00000+00	.74073-08	.00000+00	.10110-06	.00000+00	.74073-08	.00000+00	1
	.16251-07	.00000+00	.16729-08	.00000+00	.16251-07	.00000+00	.16729-08	.00000+00	2
	.33787-08	.00000+00	.43677-09	.00000+00	.33787-08	.00000+00	.43677-09	.00000+00	3
19	.00000+00	.15267-06	.00000+00	.34322-08	.00000+00	.15267-06	.00000+00	.34322-08	0
	.00000+00	.21369-07	.00000+00	.74051-09	.00000+00	.21369-07	.00000+00	.74052-09	1
	.00000+00	.39556-08	.00000+00	.18522-09	.00000+00	.39556-08	.00000+00	.18524-09	2
	.00000+00	.88567-09	.00000+00	.52076-10	.00000+00	.88569-09	.00000+00	.52085-10	3
20	.42737-06	.00000+00	.16830-07	.00000+00	.42737-06	.00000+00	.16830-07	.00000+00	0
	.39556-07	.00000+00	.28983-08	.00000+00	.39556-07	.00000+00	.28983-08	.00000+00	1
	.59044-08	.00000+00	.60781-09	.00000+00	.59046-08	.00000+00	.60785-09	.00000+00	2
	.11432-08	.00000+00	.14778-09	.00000+00	.11433-08	.00000+00	.14780-09	.00000+00	3
21	.00000+00	.66025-07	.00000+00	.14843-08	.00000+00	.66026-07	.00000+00	.14843-08	0
	.00000+00	.85807-08	.00000+00	.29736-09	.00000+00	.85812-08	.00000+00	.29732-09	1
	.00000+00	.14793-08	.00000+00	.69268-10	.00000+00	.14794-08	.00000+00	.69272-10	2
	.00000+00	.30927-09	.00000+00	.18185-10	.00000+00	.30933-09	.00000+00	.18223-10	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
22	.18878-06	.00000+00	.74342-08	.00000+00	.18879-06	.00000+00	.74346-08	.00000+00	0
	.16272-07	.00000+00	.11923-08	.00000+00	.16273-07	.00000+00	.11924-08	.00000+00	1
	.22680-08	.00000+00	.23347-09	.00000+00	.22685-08	.00000+00	.23359-09	.00000+00	2
	.41105-09	.00000+00	.53137-10	.00000+00	.41130-09	.00000+00	.53099-10	.00000+00	3
23	.00000+00	.29747-07	.00000+00	.66873-09	.00000+00	.29750-07	.00000+00	.66885-09	0
	.00000+00	.36100-08	.00000+00	.12510-09	.00000+00	.36110-08	.00000+00	.12579-09	1
	.00000+00	.58255-09	.00000+00	.27279-10	.00000+00	.58285-09	.00000+00	.27735-10	2
	.00000+00	.11427-09	.00000+00	.67191-11	.00000+00	.11442-09	.00000+00	.72467-11	3
24	.86641-07	.00000+00	.34119-08	.00000+00	.86659-07	.00000+00	.34133-08	.00000+00	0
	.69906-08	.00000+00	.51221-09	.00000+00	.69943-08	.00000+00	.51295-09	.00000+00	1
	.91418-09	.00000+00	.94107-10	.00000+00	.91526-09	.00000+00	.94761-10	.00000+00	2
	.15579-09	.00000+00	.20139-10	.00000+00	.15596-09	.00000+00	.21214-10	.00000+00	3
25	.00000+00	.13891-07	.00000+00	.31228-09	.00000+00	.13901-07	.00000+00	.31383-09	0
	.00000+00	.15816-08	.00000+00	.54811-10	.00000+00	.15850-08	.00000+00	.55627-10	1
	.00000+00	.23998-09	.00000+00	.11237-10	.00000+00	.24214-09	.00000+00	.12194-10	2
	.00000+00	.44350-10	.00000+00	.26077-11	.00000+00	.45585-10	.00000+00	.27586-11	3
26	.41123-07	.00000+00	.16194-08	.00000+00	.41196-07	.00000+00	.16232-08	.00000+00	0
	.31198-08	.00000+00	.22859-09	.00000+00	.31341-08	.00000+00	.23259-09	.00000+00	1
	.38437-09	.00000+00	.39567-10	.00000+00	.38782-09	.00000+00	.40462-10	.00000+00	2
	.61825-10	.00000+00	.79922-11	.00000+00	.62130-10	.00000+00	.98447-11	.00000+00	3
27	.00000+00	.66952-08	.00000+00	.15051-09	.00000+00	.67341-08	.00000+00	.15296-09	0
	.00000+00	.71920-09	.00000+00	.24889-10	.00000+00	.73267-09	.00000+00	.26457-10	1
	.00000+00	.10286-09	.00000+00	.48164-11	.00000+00	.10763-09	.00000+00	.49751-11	2
	.00000+00	.17974-10	.00000+00	.10568-11	.00000+00	.20048-10	.00000+00	.12284-11	3

n	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	p
28	.20110-07	.00000+00	.79192-09	.00000+00	.20504-07	.00000+00	.80263-09	.00000+00	0
	.14400-08	.00000+00	.10551-09	.00000+00	.15382-08	.00000+00	.11215-09	.00000+00	1
	.16776-09	.00000+00	.17269-10	.00000+00	.21102-09	.00000+00	.17755-10	.00000+00	2
	.25556-10	.00000+00	.33036-11	.00000+00	.46939-10	.00000+00	.34135-11	.00000+00	3
29	.00000+00	.33193-08	.00000+00	.74619-10	.00000+00	.35369-08	.00000+00	.78544-10	0
	.00000+00	.33667-09	.00000+00	.11667-10	.00000+00	.42059-09	.00000+00	.13188-10	1
	.00000+00	.45667-10	.00000+00	.21384-11	.00000+00	.93413-10	.00000+00	.21765-11	2
	.00000+00	.75696-11	.00000+00	.44509-12	.00000+00	.36131-10	.00000+00	-.41349-12	3
30	.10100-07	.00000+00	.39775-09	.00000+00	.12443-07	.00000+00	.49419-09	.00000+00	0
	.68500-09	.00000+00	.50190-10	.00000+00	.11928-08	.00000+00	.96839-10	.00000+00	1
	.75696-10	.00000+00	.77922-11	.00000+00	.23262-09	.00000+00	.21914-10	.00000+00	2
	.10955-10	.00000+00	.14161-11	.00000+00	.85393-10	.00000+00	.84406-11	.00000+00	3
31	.00000+00	.16878-08	.00000+00	.37943-10	.00000+00	.30002-08	.00000+00	.67807-10	0
	.00000+00	.16239-09	.00000+00	.56277-11	.00000+00	.55588-09	.00000+00	.20309-10	1
	.00000+00	.20925-10	.00000+00	.97985-12	.00000+00	.15635-09	.00000+00	.64387-11	2
	.00000+00	.32996-11	.00000+00	.19401-12	.00000+00	.79753-10	.00000+00	-.57262-12	3
32	.51966-08	.00000+00	.20464-09	.00000+00	.17885-07	.00000+00	.67983-09	.00000+00	0
	.33480-09	.00000+00	.24531-10	.00000+00	.27394-08	.00000+00	.17180-09	.00000+00	1
	.35195-10	.00000+00	.36230-11	.00000+00	.74949-09	.00000+00	.55763-10	.00000+00	2
	.48516-11	.00000+00	.62718-12	.00000+00	.20007-09	.00000+00	.20067-10	.00000+00	3
33	.00000+00	.87818-09	.00000+00	.19742-10	.00000+00	.66631-08	.00000+00	.23110-09	0
	.00000+00	.80378-10	.00000+00	.27854-11	.00000+00	.12310-08	.00000+00	.12889-09	1
	.00000+00	.98654-11	.00000+00	.46196-12	.00000+00	.16274-09	.00000+00	.49957-10	2
	.00000+00	.14836-11	.00000+00	.87234-13	.00000+00	-.11134-09	.00000+00	.71864-10	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	P
34	.27328-08	.00000+00	.10762-09	.00000+00	.41523-07	.00000+00	.18382-08	.00000+00	0
	.16771-09	.00000+00	.12288-10	.00000+00	.23404-08	.00000+00	.33168-09	.00000+00	1
	.16814-10	.00000+00	.17309-11	.00000+00	-.15221-08	.00000+00	.10800-10	.00000+00	2
	.22131-11	.00000+00	.28610-12	.00000+00	-.16891-08	.00000+00	-.10554-09	.00000+00	3
35	.00000+00	.46656-09	.00000+00	.10489-10	.00000+00	.63507-08	.00000+00	.46468-09	0
	.00000+00	.40727-10	.00000+00	.14113-11	.00000+00	-.37873-08	.00000+00	.31782-10	1
	.00000+00	.47729-11	.00000+00	.22350-12	.00000+00	-.40069-08	.00000+00	-.29783-10	2
	.00000+00	.68613-12	.00000+00	.40344-13	.00000+00	-.28558-08	.00000+00	-.84523-10	3
36	.14662-08	.00000+00	.57738-10	.00000+00	-.13268-06	.00000+00	-.51996-08	.00000+00	0
	.85913-10	.00000+00	.62949-11	.00000+00	-.67032-07	.00000+00	-.50277-08	.00000+00	1
	.82336-11	.00000+00	.84757-12	.00000+00	-.34090-07	.00000+00	-.36140-08	.00000+00	2
	.10371-11	.00000+00	.13406-12	.00000+00	-.18937-07	.00000+00	-.25154-08	.00000+00	3
37	.00000+00	.25266-09	.00000+00	.56800-11	.00000+00	-.19739-06	.00000+00	-.50544-08	0
	.00000+00	.21083-10	.00000+00	.73060-12	.00000+00	-.87469-07	.00000+00	-.35211-08	1
	.00000+00	.23644-11	.00000+00	.11071-12	.00000+00	-.42924-07	.00000+00	-.30923-08	2
	.00000+00	.32558-12	.00000+00	.19144-13	.00000+00	-.23369-07	.00000+00	-.22977-08	3
38	.80114-09	.00000+00	.31549-10	.00000+00	-.33176-05	.00000+00	-.13333-06	.00000+00	0
	.44923-10	.00000+00	.32915-11	.00000+00	-.81760-06	.00000+00	-.62282-07	.00000+00	1
	.41240-11	.00000+00	.42452-12	.00000+00	-.29549-06	.00000+00	-.32823-07	.00000+00	2
	.49805-12	.00000+00	.64383-13	.00000+00	-.12949-06	.00000+00	-.18833-07	.00000+00	3
39	.00000+00	.13925-09	.00000+00	.31305-11	.00000+00	-.25499-05	.00000+00	-.61687-07	0
	.00000+00	.11131-10	.00000+00	.38572-12	.00000+00	-.80893-06	.00000+00	-.31298-07	1
	.00000+00	.11969-11	.00000+00	.56044-13	.00000+00	-.32015-06	.00000+00	-.21731-07	2
	.00000+00	.15817-12	.00000+00	.93003-14	.00000+00	-.14752-06	.00000+00	-.12505-07	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
40	.44522-09	.00000+00	.17533-10	.00000+00	-.32334-04	.00000+00	-.12807-05	.00000+00	0
	.23937-10	.00000+00	.17539-11	.00000+00	-.64025-05	.00000+00	-.47239-06	.00000+00	1
	.21089-11	.00000+00	.21710-12	.00000+00	-.19642-05	.00000+00	-.20446-06	.00000+00	2
	.24465-12	.00000+00	.31626-13	.00000+00	-.75947-06	.00000+00	-.99418-07	.00000+00	3

TABLE II

Angular-spatial-pathlength moments as determined by Eqs. (13) and (50) with a plane isotropic source of 0.1 MeV electrons

	$\phi^{p}_{\ell n}$								
	From Eq. (50 Double Precision or from Eq. (13)				From Eq. (50) Single Precision				
n	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	p
	.10000+01	.00000+00	.00000+00	.00000+00	.10000+01	.00000+00	.00000+00	.00000+00	0
0	.50000+00	.00000+00	.00000+00	.00000+00	.50000+00	.00000+00	.00000+00	.00000+00	1
	.33333+00	.00000+00	.00000+00	.00000+00	.33333+00	.00000+00	.00000+00	.00000+00	2
	.25000+00	.00000+00	.00000+00	.00000+00	.25000+00	.00000+00	.00000+00	.00000+00	3
	.00000+00	.40791-01	.00000+00	.00000+00	.00000+00	.40791-01	.00000+00	.00000+00	0
1	.00000+00	.21847-01	.00000+00	.00000+00	.00000+00	.21847-01	.00000+00	.00000+00	1
	.00000+00	.13693-01	.00000+00	.00000+00	.00000+00	.13693-01	.00000+00	.00000+00	2
	.00000+00	.94084-02	.00000+00	.00000+00	.00000+00	.94084-02	.00000+00	.00000+00	3
	.43694-01	.00000+00	.18833-02	.00000+00	.43694-01	.00000+00	.18833-02	.00000+00	0
2	.13693-01	.00000+00	.10655-02	.00000+00	.13693-01	.00000+00	.10655-02	.00000+00	1
	.62723-02	.00000+00	.66723-03	.00000+00	.62723-02	.00000+00	.66723-03	.00000+00	2
	.34354-02	.00000+00	.44758-03	.00000+00	.34354-02	.00000+00	.44758-03	.00000+00	3
	.00000+00	.38729-02	.00000+00	.84259-04	.00000+00	.38729-02	.00000+00	.84259-04	0
3	.00000+00	.14957-02	.00000+00	.49705-04	.00000+00	.14957-02	.00000+00	.49705-04	1
	.00000+00	.71157-03	.00000+00	.31516-04	.00000+00	.71157-03	.00000+00	.31516-04	2
	.00000+00	.38497-03	.00000+00	.21076-04	.00000+00	.38497-03	.00000+00	.21076-04	3

n	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	p
4	.59827-02	.00000+00	.27071-03	.00000+00	.59827-02	.00000+00	.27071-03	.00000+00	0
	.14231-02	.00000+00	.11810-03	.00000+00	.14231-02	.00000+00	.11810-03	.00000+00	1
	.51329-03	.00000+00	.59087-04	.00000+00	.51329-03	.00000+00	.59087-04	.00000+00	2
	.22755-03	.00000+00	.32517-04	.00000+00	.22755-03	.00000+00	.32517-04	.00000+00	3
5	.00000+00	.67687-03	.00000+00	.15947-04	.00000+00	.67687-03	.00000+00	.15947-04	0
	.00000+00	.20694-03	.00000+00	.75675-05	.00000+00	.20694-03	.00000+00	.75675-05	1
	.00000+00	.80127-04	.00000+00	.39631-05	.00000+00	.80127-04	.00000+00	.39631-05	2
	.00000+00	.36049-04	.00000+00	.22340-05	.00000+00	.36049-04	.00000+00	.22340-05	3
6	.12416-02	.00000+00	.56450-04	.00000+00	.12416-02	.00000+00	.56450-04	.00000+00	0
	.24038-03	.00000+00	.20093-04	.00000+00	.24038-03	.00000+00	.20093-04	.00000+00	1
	.72099-04	.00000+00	.83826-05	.00000+00	.72099-04	.00000+00	.83826-05	.00000+00	2
	.27043-04	.00000+00	.39143-05	.00000+00	.27043-04	.00000+00	.39143-05	.00000+00	3
7	.00000+00	.16023-03	.00000+00	.38100-05	.00000+00	.16023-03	.00000+00	.38100-05	0
	.00000+00	.40770-04	.00000+00	.15097-05	.00000+00	.40770-04	.00000+00	.15097-05	1
	.00000+00	.13370-04	.00000+00	.67185-06	.00000+00	.13370-04	.00000+00	.67185-06	2
	.00000+00	.51675-05	.00000+00	.32647-06	.00000+00	.51675-05	.00000+00	.32647-06	3
8	.32616-03	.00000+00	.14839-04	.00000+00	.32616-03	.00000+00	.14839-04	.00000+00	0
	.53479-04	.00000+00	.44752-05	.00000+00	.53479-04	.00000+00	.44752-05	.00000+00	1
	.13780-04	.00000+00	.16048-05	.00000+00	.13780-04	.00000+00	.16048-05	.00000+00	2
	.44936-05	.00000+00	.65192-06	.00000+00	.44936-05	.00000+00	.65192-06	.00000+00	3
9	.00000+00	.45838-04	.00000+00	.10916-05	.00000+00	.45838-04	.00000+00	.10916-05	0
	.00000+00	.10022-04	.00000+00	.37189-06	.00000+00	.10022-04	.00000+00	.37189-06	1
	.00000+00	.28578-05	.00000+00	.14403-06	.00000+00	.28578-05	.00000+00	.14403-06	2
	.00000+00	.97035-06	.00000+00	.61535-07	.00000+00	.97035-06	.00000+00	.61535-07	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
10	.10022-03	.00000+00	.45599-05	.00000+00	.10022-03	.00000+00	.45599-05	.00000+00	0
	.14289-04	.00000+00	.11960-05	.00000+00	.14289-04	.00000+00	.11960-05	.00000+00	1
	.32345-05	.00000+00	.37681-06	.00000+00	.32345-05	.00000+00	.37681-06	.00000+00	2
	.93471-06	.00000+00	.13567-06	.00000+00	.93471-06	.00000+00	.13567-06	.00000+00	3
11	.00000+00	.14970-04	.00000+00	.35658-06	.00000+00	.14970-04	.00000+00	.35658-06	0
	.00000+00	.28752-05	.00000+00	.10674-06	.00000+00	.28752-05	.00000+00	.10674-06	1
	.00000+00	.72664-06	.00000+00	.36642-07	.00000+00	.72664-06	.00000+00	.36642-07	2
	.00000+00	.22032-06	.00000+00	.13983-07	.00000+00	.22032-06	.00000+00	.13983-07	3
12	.34503-04	.00000+00	.15699-05	.00000+00	.34503-04	.00000+00	.15699-05	.00000+00	0
	.43598-05	.00000+00	.36493-06	.00000+00	.43598-05	.00000+00	.36493-06	.00000+00	1
	.88130-06	.00000+00	.10268-06	.00000+00	.88130-06	.00000+00	.10268-06	.00000+00	2
	.22896-06	.00000+00	.33236-07	.00000+00	.22896-06	.00000+00	.33236-07	.00000+00	3
13	.00000+00	.53980-05	.00000+00	.12858-06	.00000+00	.53980-05	.00000+00	.12858-06	0
	.00000+00	.92587-06	.00000+00	.34375-07	.00000+00	.92587-06	.00000+00	.34375-07	1
	.00000+00	.21036-06	.00000+00	.10609-07	.00000+00	.21036-06	.00000+00	.10609-07	2
	.00000+00	.57682-07	.00000+00	.36617-08	.00000+00	.57682-07	.00000+00	.36617-08	3
14	.12962-04	.00000+00	.58979-06	.00000+00	.12962-04	.00000+00	.58979-06	.00000+00	0
	.14725-05	.00000+00	.12325-06	.00000+00	.14725-05	.00000+00	.12325-06	.00000+00	1
	.26918-06	.00000+00	.31362-07	.00000+00	.26918-06	.00000+00	.31362-07	.00000+00	2
	.63579-07	.00000+00	.92296-08	.00000+00	.63579-07	.00000+00	.92296-08	.00000+00	3
15	.00000+00	.21036-05	.00000+00	.50111-07	.00000+00	.21036-05	.00000+00	.50111-07	0
	.00000+00	.32630-06	.00000+00	.12115-07	.00000+00	.32630-06	.00000+00	.12115-07	1
	.00000+00	.67401-07	.00000+00	.33995-08	.00000+00	.67401-07	.00000+00	.33995-08	2
	.00000+00	.16883-07	.00000+00	.10718-08	.00000+00	.16883-07	.00000+00	.10718-08	3

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16	.52209-05	.00000+00	.23756-06	.00000+00	.52209-05	.00000+00	.23756-06	.00000+00	0
	.53921-06	.00000+00	.45134-07	.00000+00	.53921-06	.00000+00	.45134-07	.00000+00	1
	.90042-07	.00000+00	.10491-07	.00000+00	.90042-07	.00000+00	.10491-07	.00000+00	2
	.19511-07	.00000+00	.28323-08	.00000+00	.19510-07	.00000+00	.28323-08	.00000+00	3
17	.00000+00	.87302-06	.00000+00	.20796-07	.00000+00	.87302-06	.00000+00	.20796-07	0
	.00000+00	.12370-06	.00000+00	.45928-08	.00000+00	.12370-06	.00000+00	.45927-08	1
	.00000+00	.23441-07	.00000+00	.11823-08	.00000+00	.23441-07	.00000+00	.11822-08	2
	.00000+00	.54076-08	.00000+00	.34330-09	.00000+00	.54074-08	.00000+00	.34311-09	3
18	.22266-05	.00000+00	.10132-06	.00000+00	.22266-05	.00000+00	.10131-06	.00000+00	0
	.21097-06	.00000+00	.17659-07	.00000+00	.21097-06	.00000+00	.17659-07	.00000+00	1
	.32446-07	.00000+00	.37802-08	.00000+00	.32444-07	.00000+00	.37800-08	.00000+00	2
	.64978-08	.00000+00	.94327-09	.00000+00	.64971-08	.00000+00	.94315-09	.00000+00	3
19	.00000+00	.38176-06	.00000+00	.90941-08	.00000+00	.38176-06	.00000+00	.90939-08	0
	.00000+00	.49819-07	.00000+00	.18497-08	.00000+00	.49817-07	.00000+00	.18502-08	1
	.00000+00	.87253-08	.00000+00	.44008-09	.00000+00	.87241-08	.00000+00	.44019-09	2
	.00000+00	.18664-08	.00000+00	.11849-09	.00000+00	.18657-08	.00000+00	.11859-09	3
20	.99638-06	.00000+00	.45337-07	.00000+00	.99633-06	.00000+00	.45335-07	.00000+00	0
	.87253-07	.00000+00	.73034-08	.00000+00	.87241-07	.00000+00	.73029-08	.00000+00	1
	.12443-07	.00000+00	.14497-08	.00000+00	.12438-07	.00000+00	.14504-08	.00000+00	2
	.23175-08	.00000+00	.33642-09	.00000+00	.23147-08	.00000+00	.33642-09	.00000+00	3
21	.00000+00	.17451-06	.00000+00	.41571-08	.00000+00	.17449-06	.00000+00	.41557-08	0
	.00000+00	.21116-07	.00000+00	.78400-09	.00000+00	.21109-07	.00000+00	.78065-09	1
	.00000+00	.34395-08	.00000+00	.17348-09	.00000+00	.34372-08	.00000+00	.17098-09	2
	.00000+00	.68612-09	.00000+00	.43558-10	.00000+00	.68558-09	.00000+00	.40742-10	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
22	.46455-06	.00000+00	.21138-07	.00000+00	.46440-06	.00000+00	.21130-07	.00000+00	0
	.37834-07	.00000+00	.31669-08	.00000+00	.37806-07	.00000+00	.31675-08	.00000+00	1
	.50315-08	.00000+00	.58621-09	.00000+00	.50234-08	.00000+00	.58703-09	.00000+00	2
	.87617-09	.00000+00	.12719-09	.00000+00	.87296-09	.00000+00	.13009-09	.00000+00	3
23	.00000+00	.82876-07	.00000+00	.19742-08	.00000+00	.82807-07	.00000+00	.19753-08	0
	.00000+00	.93521-08	.00000+00	.34723-09	.00000+00	.93307-08	.00000+00	.34499-09	1
	.00000+00	.14242-08	.00000+00	.71833-10	.00000+00	.14229-08	.00000+00	.68414-10	2
	.00000+00	.26626-09	.00000+00	.16903-10	.00000+00	.26366-09	.00000+00	.12193-10	3
24	.22445-06	.00000+00	.10213-07	.00000+00	.22397-06	.00000+00	.10214-07	.00000+00	0
	.17091-07	.00000+00	.14306-08	.00000+00	.16992-07	.00000+00	.14390-08	.00000+00	1
	.21301-08	.00000+00	.24817-09	.00000+00	.20817-08	.00000+00	.24341-09	.00000+00	2
	.34838-09	.00000+00	.50574-10	.00000+00	.32742-09	.00000+00	.78334-10	.00000+00	3
25	.00000+00	.40693-07	.00000+00	.96936-09	.00000+00	.40447-07	.00000+00	.93493-09	0
	.00000+00	.43034-08	.00000+00	.15978-09	.00000+00	.42189-08	.00000+00	.15009-09	1
	.00000+00	.61554-09	.00000+00	.31046-10	.00000+00	.54868-09	.00000+00	.20882-10	2
	.00000+00	.10830-09	.00000+00	.68757-11	.00000+00	.66334-10	.00000+00	-.88701-11	3
26	.11189-06	.00000+00	.50911-08	.00000+00	.10912-06	.00000+00	.49312-08	.00000+00	0
	.80020-08	.00000+00	.66980-09	.00000+00	.71353-08	.00000+00	.59995-09	.00000+00	1
	.93864-09	.00000+00	.10936-09	.00000+00	.52225-09	.00000+00	.48675-10	.00000+00	2
	.14476-09	.00000+00	.21015-10	.00000+00	-.14081-09	.00000+00	-.22358-10	.00000+00	3
27	.00000+00	.20577-07	.00000+00	.49017-09	.00000+00	.18517-07	.00000+00	.36400-09	0
	.00000+00	.20481-08	.00000+00	.76041-10	.00000+00	.12679-08	.00000+00	.61615-11	1
	.00000+00	.27624-09	.00000+00	.13933-10	.00000+00	-.16169-09	.00000+00	-.69626-10	2
	.00000+00	.45915-10	.00000+00	.29149-11	.00000+00	-.19147-09	.00000+00	-.61642-10	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
28	.57346-07	.00000+00	.26093-08	.00000+00	.36134-07	.00000+00	.15728-08	.00000+00	0
	.38674-08	.00000+00	.32371-09	.00000+00	-.87570-09	.00000+00	-.25801-09	.00000+00	1
	.42854-09	.00000+00	.49928-10	.00000+00	-.96838-09	.00000+00	-.17999-09	.00000+00	2
	.62542-10	.00000+00	.90791-11	.00000+00	-.33407-09	.00000+00	-.17888-09	.00000+00	3
29	.00000+00	.10681-07	.00000+00	.25445-09	.00000+00	-.39258-08	.00000+00	-.24117-09	0
	.00000+00	.10043-08	.00000+00	.37289-10	.00000+00	-.34275-08	.00000+00	-.25551-09	1
	.00000+00	.12818-09	.00000+00	.64651-11	.00000+00	-.14615-08	.00000+00	-.14931-09	2
	.00000+00	.20193-10	.00000+00	.12820-11	.00000+00	-.81875-09	.00000+00	-.11447-09	3
30	.30130-07	.00000+00	.13709-08	.00000+00	-.10354-06	.00000+00	-.43597-08	.00000+00	0
	.19227-08	.00000+00	.16094-09	.00000+00	-.23228-07	.00000+00	-.18947-08	.00000+00	1
	.20193-09	.00000+00	.23526-10	.00000+00	-.67634-08	.00000+00	-.62452-09	.00000+00	2
	.27973-10	.00000+00	.40608-11	.00000+00	-.22420-08	.00000+00	-.16165-09	.00000+00	3
31	.00000+00	.56768-08	.00000+00	.13523-09	.00000+00	-.65256-07	.00000+00	-.18353-08	0
	.00000+00	.50588-09	.00000+00	.18782-10	.00000+00	-.14969-07	.00000+00	-.50223-09	1
	.00000+00	.61286-10	.00000+00	.30911-11	.00000+00	-.26768-08	.00000+00	-.39029-09	2
	.00000+00	.91771-11	.00000+00	.58261-12	.00000+00	-.50621-10	.00000+00	-.15774-09	3
32	.16188-07	.00000+00	.73658-09	.00000+00	-.46195-06	.00000+00	-.22162-07	.00000+00	0
	.98057-09	.00000+00	.82078-10	.00000+00	-.43859-07	.00000+00	-.44614-08	.00000+00	1
	.97889-10	.00000+00	.11405-10	.00000+00	.26496-08	.00000+00	-.29639-09	.00000+00	2
	.12907-10	.00000+00	.18737-11	.00000+00	.98789-08	.00000+00	.75745-09	.00000+00	3
33	.00000+00	.30819-08	.00000+00	.73414-10	.00000+00	-.13919-06	.00000+00	-.32986-08	0
	.00000+00	.26106-09	.00000+00	.96925-11	.00000+00	.10156-07	.00000+00	.11804-09	1
	.00000+00	.30102-10	.00000+00	.15183-11	.00000+00	.23195-07	.00000+00	.63032-09	2
	.00000+00	.42959-11	.00000+00	.27272-12	.00000+00	.19785-07	.00000+00	.11950-08	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	P
34	.88759-08	.00000+00	.40386-09	.00000+00	.30852-06	.00000+00	.13962-07	.00000+00	0
	.51174-09	.00000+00	.42835-10	.00000+00	.38069-06	.00000+00	.32669-07	.00000+00	1
	.48687-10	.00000+00	.56724-11	.00000+00	.21277-06	.00000+00	.24723-07	.00000+00	2
	.61254-11	.00000+00	.88921-12	.00000+00	.12080-06	.00000+00	.17453-07	.00000+00	3
35	.00000+00	.17058-08	.00000+00	.40635-10	.00000+00	.12723-05	.00000+00	.25552-07	0
	.00000+00	.13771-09	.00000+00	.51129-11	.00000+00	.60736-06	.00000+00	.20561-07	1
	.00000+00	.15152-10	.00000+00	.76421-12	.00000+00	.30204-06	.00000+00	.13737-07	2
	.00000+00	.20655-11	.00000+00	.13113-12	.00000+00	.16562-06	.00000+00	.81014-08	3
36	.49575-08	.00000+00	.22557-09	.00000+00	.21802-04	.00000+00	.98716-06	.00000+00	0
	.27273-09	.00000+00	.22829-10	.00000+00	.54198-05	.00000+00	.44971-06	.00000+00	1
	.24787-10	.00000+00	.28879-11	.00000+00	.19471-05	.00000+00	.22597-06	.00000+00	2
	.29823-11	.00000+00	.43293-12	.00000+00	.84444-06	.00000+00	.11911-06	.00000+00	3
37	.00000+00	.96107-09	.00000+00	.22894-10	.00000+00	.19069-04	.00000+00	.43344-06	0
	.00000+00	.74116-10	.00000+00	.27518-11	.00000+00	.57904-05	.00000+00	.18551-06	1
	.00000+00	.77985-11	.00000+00	.39333-12	.00000+00	.21859-05	.00000+00	.87219-07	2
	.00000+00	.10178-11	.00000+00	.64613-13	.00000+00	.95568-06	.00000+00	.45977-07	3
38	.28164-08	.00000+00	.12815-09	.00000+00	.22008-03	.00000+00	.99947-05	.00000+00	0
	.14817-09	.00000+00	.12403-10	.00000+00	.41555-04	.00000+00	.34755-05	.00000+00	1
	.12892-10	.00000+00	.15020-11	.00000+00	.12048-04	.00000+00	.13880-05	.00000+00	2
	.14863-11	.00000+00	.21577-12	.00000+00	.43477-05	.00000+00	.61516-06	.00000+00	3
39	.00000+00	.55036-09	.00000+00	.13110-10	.00000+00	.15407-03	.00000+00	.37041-05	0
	.00000+00	.40631-10	.00000+00	.15086-11	.00000+00	.37808-04	.00000+00	.14582-05	1
	.00000+00	.40968-11	.00000+00	.20663-12	.00000+00	.11845-04	.00000+00	.71305-06	2
	.00000+00	.51283-12	.00000+00	.32557-13	.00000+00	.42694-05	.00000+00	.33009-06	3

n	ℓ=0	ℓ=1	ℓ=2	ℓ=3	ℓ=0	ℓ=1	ℓ=2	ℓ=3	p
40	.16252-08	.00000+00	.73951-10	.00000+00	.15119-02	.00000+00	.68962-04	.00000+00	0
	.81935-10	.00000+00	.68583-11	.00000+00	.23668-03	.00000+00	.19921-04	.00000+00	1
	.68377-11	.00000+00	.79665-12	.00000+00	.57419-04	.00000+00	.68714-05	.00000+00	2
	.75684-12	.00000+00	.10987-12	.00000+00	.17220-04	.00000+00	.26428-05	.00000+00	3

TABLE III

Angular-spatial-pathlength moments, $M^{p}_{\ell n}$, $\ell=0$, with cross sections given by Eq. (46), ($\alpha<\infty$) and the corresponding values for Eq. (11), ($\alpha=\infty$). The $\alpha<\infty$ values were taken from Ref. [10].

Carbon 0.025 MeV electrons

$\alpha = -107.8$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	0.200623E 01	0.665423E 00	0.398403E-00	0.284237E-00	0.220907E-00	0.180646E-00	0.152797E-00
2	0.207134E-00	0.353090E-01	0.128270E-01	0.613529E-02	0.341688E-02	0.229981E-02	0.138322E-02
4	0.554574E-01	0.665209E-02	0.179023E-02	0.659319E-03	0.291312E-03	0.149455E-03	0.793759E-04
6	0.203295E-01	0.190310E-02	0.410774E-03	0.124023E-03	0.457380E-04	0.193494E-04	0.908069E-05
8	0.882493E-02	0.680553E-03	0.123130E-03	0.316172E-04	0.100391E-04	0.369538E-05	0.151946E-05
10	0.427186E-02	0.280751E-03	0.438166E-04	0.980682E-05	0.273872E-05	0.893686E-06	0.328035E-06
12	0.223431E-02	0.128119E-03	0.176028E-04	0.349565E-05	0.872190E-06	0.255850E-06	0.848891E-07

$\alpha = \infty$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	.200000E+01	.666667E+00	.400000E+00	.285714E+00	.222222E+00	.181818E+00	.153846E+00
2	.205920E+00	.353716E-01	.129105E-01	.619536E-02	.345333E-02	.212933E-02	.140470E-02
4	.550120E-01	.665703E-02	.180250E-02	.656952E-03	.295755E-03	.145138E-03	.313230E-04
6	.201279E-01	.190215E-02	.413483E-03	.125540E-03	.465093E-04	.197539E-04	.927874E-05
8	.972232E-02	.679354E-03	.123865E-03	.320068E-04	.102157E-04	.377779E-05	.155947E-05
10	.421551E-02	.279904E-03	.440439E-04	.992545E-05	.279815E-05	.314469E-06	.337173E-06
12	.220159E-02	.127575E-03	.176790E-04	.353647E-05	.887984E-06	.261942E-06	.873434E-07

Aluminum 0.1 MeV electrons

$\alpha = 35.45$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	0.198161E 01	0.670368E 00	0.404776E-00	0.290147E-00	0.226177E-00	0.185342E-00	0.157008E-00
2	0.144022E-00	0.278011E-01	0.107991E-01	0.540427E-02	0.310972E-02	0.195892E-02	0.131968E-02
4	0.288370E-01	0.401017E-02	0.119195E-02	0.472056E-03	0.220571E-03	0.115158E-03	0.651802E-04
6	0.807240E-02	0.894246E-03	0.217362E-03	0.718395E-04	0.284736E-04	0.127806E-04	0.528995E-05
8	0.274594E-02	0.253993E-03	0.524374E-04	0.149249E-04	0.515314E-05	0.203446E-05	0.888262E-06
10	0.106227E-02	0.845683E-04	0.152087E-04	0.380881E-05	0.116707E-05	0.411990E-06	0.161857E-06
12	0.450951E-03	0.315657E-04	0.503610E-05	0.112746E-05	0.310888E-06	0.993416E-07	0.359116E-07

$\alpha = \infty$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	.200000E+01	.666667E+00	.400000E+00	.285714E+00	.222222E+00	.181818E+00	.153846E+00
2	.148727E+00	.276595E-01	.106085E-01	.525843E-02	.300449E-02	.198232E-02	.125875E-02
4	.295995E-01	.401129E-02	.117107E-02	.457552E-03	.211525E-03	.109452E-03	.615226E-04
6	.834394E-02	.898370E-03	.214005E-03	.695811E-04	.272113E-04	.120776E-04	.597731E-05
8	.285662E-02	.256582E-03	.517818E-04	.144676E-04	.491914E-05	.191619E-05	.926922E-05
10	.111174E-02	.858527E-04	.150695E-04	.369841E-05	.111390E-05	.387314E-06	.150150E-05
12	.474626E-03	.322000E-04	.500815E-05	.109723E-05	.296934E-06	.933293E-07	.329756E-07

Aluminum 2.0 MeV electrons

$\alpha = 1.288$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	0.167676E 01	0.739576E 00	0.500452E 00	0.383056E-00	0.311804E-00	0.263525E-00	0.228488E-00
2	0.142005E-00	0.360462E-01	0.161241E-01	0.104595E-01	0.670000E-02	0.459232E-02	0.330322E-02
4	0.317096E-01	0.645579E-02	0.248215E-02	0.119390E-02	0.651216E-03	0.386159E-03	0.243305E-03
6	0.961214E-02	0.163376E-02	0.533581E-03	0.222737E-03	0.107011E-03	0.565199E-04	0.320015E-04
8	0.363200E-02	0.514562E-03	0.146902E-03	0.544688E-04	0.235091E-04	0.112521E-04	0.581382E-05
10	0.151146E-02	0.166986E-03	0.475634E-04	0.159122E-04	0.625337E-05	0.274463E-05	0.130794E-05
12	0.664819E-03	0.753012E-04	0.173055E-04	0.528451E-05	0.190997E-05	0.775567E-06	0.343609E-06

$\alpha = \infty$

n	p = −1/2	1/2	3/2	5/2	7/2	9/2	11/2
0	.200000E+01	.666667E+00	.400000E+00	.285714E+00	.222222E+00	.181818E+00	.153846E+00
2	.133333E+00	.344224E-01	.125377E-01	.603754E-02	.340733E-02	.210191E-02	.139569E-02
4	.304029E-01	.519323E-02	.153433E-02	.631540E-03	.281737E-03	.141747E-03	.779384E-04
6	.175383E-01	.183415E-02	.370779E-03	.113713E-03	.424733E-04	.181593E-04	.858070E-05
8	.723972E-02	.572086E-03	.135906E-03	.275574E-04	.891430E-05	.332377E-05	.133207E-05
10	.311193E-02	.224310E-03	.359337E-04	.917876E-05	.232235E-05	.758992E-06	.285974E-06
12	.164618E-02	.973921E-04	.137129E-04	.278002E-05	.706159E-06	.210444E-06	.708170E-07

$= 0$, $p = -\frac{1}{2}, \frac{1}{2}, \ldots \frac{11}{2}$ and three different values of α. The case $\alpha = 1.288$ corresponds to 2.0 MeV electrons in aluminum, $\alpha = 35.45$ corresponds to 0.1 MeV electrons in aluminum, and $\alpha = 107.8$ corresponds to 0.025 MeV electrons in carbon. As expected, the agreement improves as $|\alpha|$ increases.

IV. C. Reconstructed Electron Densities

The electron density $\phi_0(x,s)$ was reconstructed from the $\phi_{0n}(s)$, $n = 0, 1, 2, \ldots$ N for 1.0 MeV and 0.1 MeV isotropic electron sources in aluminum. All three methods described in Section II. C. were used. For small N values the first two methods give better results since there was no need to represent the rapidly varying density, $\phi_0(x,s)$ across the wave front. On the other hand, with increasing N the third method appears superior at least for small s values. As $s \to 0$ the monomials x^n, $n = 0, 1, 2, \ldots$ N begin "loosing" their linear independence on the interval $[-s,s]$, that is, the matrix defined by Eq. (41) becomes more ill conditioned as $s \to 0$. Although a special matrix inversion routine developed for ill conditioned matrices was used, it eventually broke down for s small enough or N large enough.

No difficulties were experienced inverting the matrix defined by Eq. (44). However, method 3 was subject to round off errors in the determination of the $\phi_{0n}(s)$. The best results for the 1.0 MeV source electrons were obtained with N = 34 while

the best results for the 0.1 MeV source occurred with N = 30. This conclusion is based on qualitative observations of the results for larger N values, where the influence of numerical errors is quite apparent. No independent solutions are presently available to make quantitative comparisons. It is planned to repeat these calculations using the streaming ray numerical technique so that an independent verification will be possible.

The results of the electron density calculations are shown in Tables IV-IX. The agreement between the three methods is quite good for large s although the breakdown of methods 1 and 2 at small s values is apparent. This breakdown could be avoided by decreasing N with s. In fact, with the right choice of N, methods 1 and 2 should do at least as well as method 3 since there is no need to numerically represent $\phi_0(x,s) = 0$ in the region beyond the wave front.

The electron densities determined by method 3 for plane isotropic sources in aluminum are plotted in Figs. 1 and 2 for $E_0 = 0.1$ MeV and in Figs. 3 and 4 for $E_0 = 1.0$ MeV. Figures 5 and 6 represent the same physical situation as Figs. 3 and 4 with the modification $\sigma(s=0) \rightarrow 1.0$ (range unit)$^{-1}$. Figures 1, 3 and 5 give $\phi_0(x,s)$ in the region $.025 \leq s \leq .975$, $.0125 \leq x \leq .988$. The oscillations in the region $|x| > s$, s small, are due to the problem of trying to represent a delta function with a

TABLE IV

Electron densities in aluminum from a 0.1 MeV plane isotropic source using N=10.

Method 1

	x=.15		x=.35		x=.55		x=.75		x=.95	s
.159+01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.374+01	-.481+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.247+01	.222+01	-.982-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.194+01	.180+01	.127+01	.312-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.155+01	.125+01	.551+00	.393-01	.000+00	.000+00	.000+00	.000+00	.000+00	
.153+01	.141+01	.116+01	.731+00	.179+00	.169-01	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.132+01	.110+01	.758+00	.343+00	.368-01	-.225-02	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.751+00	.405+00	.120+00	.832-03	-.888-02	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.744+00	.426+00	.164+00	.241-01	-.362-03	-.321-02	.000+00	
.135+01	.124+01	.102+01	.741+00	.435+00	.180+00	.359-01	.156-02	.183-02	.107-01	.95

Method 2

.159+01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.374+01	-.481+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.247+01	.222+01	-.982-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.194+01	.180+01	.127+01	.312-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.155+01	.125+01	.551+00	.393-01	.000+00	.000+00	.000+00	.000+00	.000+00	
.153+01	.141+01	.116+01	.731+00	.179+00	.169-01	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.132+01	.110+01	.758+00	.343+00	.368-01	-.225-02	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.751+00	.405+00	.120+00	.832-03	-.888-02	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.744+00	.426+00	.164+00	.241-01	-.362-03	-.321-02	.000+00	
.135+01	.124+01	.102+01	.741+00	.435+00	.180+00	.359-01	.156-02	.183-02	.107-01	.95

Method 3

.341+01	.209+01	.330+00	-.730+00	-.571+00	.238+00	.513+00	-.162+00	-.374+00	.541+00	
.305+01	.199+01	.529+00	-.426+00	-.436+00	.117+00	.357+00	-.875-01	-.263+00	.368+00	.15
.256+01	.183+01	.779+00	-.189-01	-.236+00	-.148-01	.159+00	-.757-02	-.124+00	.159+00	
.211+01	.166+01	.974+00	.335+00	-.268-01	-.807-01	.793-02	.343-01	-.188-01	.114-01	.35
.177+01	.152+01	.108+01	.571+00	.151+00	-.672-01	-.658-01	.351-01	.320-01	-.520-01	
.157+01	.141+01	.110+01	.692+00	.282+00	-.317-02	-.731-01	.158-01	.419-01	-.558-01	.55
.145+01	.133+01	.108+01	.737+00	.364+00	.721-01	-.441-01	-.177-02	.307-01	-.366-01	
.139+01	.128+01	.105+01	.745+00	.409+00	.132+00	-.713-02	-.801-02	.172-01	-.192-01	.75
.136+01	.125+01	.103+01	.742+00	.429+00	.168+00	.208-01	-.551-02	.946-02	-.107-01	
.135+01	.124+01	.112+01	.739+00	.437+00	.183+00	.344-01	-.174-02	.707-02	-.838-02	.95

TABLE V

Electron densities in aluminum from a 0.1 MeV plane isotropic source using N=20.

Method 1

x=.15		x=.35		x=.55		x=.75		x=.95		s
.283+05	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.373+01	.379-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.246+01	.223+01	.213-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.194+01	.180+01	.126+01	-.463-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.156+01	.124+01	.556+00	-.300-02	.000+00	.000+00	.000+00	.000+00	.000+00	
.152+01	.141+01	.116+01	.733+00	.174+00	.351-03	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.133+01	.110+01	.757+00	.345+00	.351-01	.641-03	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.365-02	-.449-04	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.243-01	.157-03	-.133-03	.000+00	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.380-01	.175-02	-.420-04	.145-03	.95

Method 2

x=.15		x=.35		x=.55		x=.75		x=.95		s
.823+08	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.373+01	.379-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.246+01	.223+01	.213-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.194+01	.180+01	.126+01	-.463-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.156+01	.124+01	.556+00	-.300-02	.000+00	.000+00	.000+00	.000+00	.000+00	
.152+01	.141+01	.116+01	.733+00	.174+00	.351-03	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.133+01	.110+01	.757+00	.345+00	.351-01	.641-03	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.365-02	-.449-04	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.243-01	.157-03	-.133-03	.000+00	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.380-01	.175-02	-.420-04	.145-03	.95

Method 3

x=.15		x=.35		x=.55		x=.75		x=.95		s
.535+01	.141-01	-.853+00	.798+00	-.373+00	-.147-01	.223+00	-.257+00	.558-01	.518+00	
.387+01	.129+01	-.112+00	.328-01	.249-01	-.445-01	.471-01	-.420-01	.193-01	.559-01	.15
.254+01	.201+01	.698+00	-.225+00	.830-01	.918-02	-.521-01	.567-01	-.129-01	-.108+00	
.190+01	.189+01	.111+01	.132+00	-.553-01	.348-01	-.233-01	.176-01	-.109-01	-.152-01	.35
.166+01	.158+01	.123+01	.539+00	-.803-02	-.918-02	.177-01	-.167-01	.350-02	.284-01	
.153+01	.140+01	.117+01	.723+00	.186+00	-.184-01	.104-01	-.750-02	.453-02	.625-02	.55
.144+01	.132+01	.110+01	.758+00	.339+00	.438-01	-.615-02	.352-02	-.442-03	-.447-02	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.285-02	.161-02	-.100-02	-.136-02	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.246-01	-.122-03	-.824-04	.828-04	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.380-01	.170-02	.462-05	.306-04	.95

TABLE VI

Electron densities in aluminum from a 0.1 MeV plane isotropic source using N=20.

x=.15		x=.35		x=.55		x=.75		x=.95		s
					Method 1					
.283+05	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
-.115+06	-.786+06	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
-.265+05	-.211+05	.184+06	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.235+01	.104+01	.294+00	.860+01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.156+01	.124+01	.556+00	.575-02	.000+00	.000+00	.000+00	.000+00	.000+00	
.152+01	.141+01	.116+01	.733+00	.174+00	-.567-04	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.133+01	.110+01	.757+00	.344+00	.350-01	-.308-04	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.368-02	.160-04	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.242-01	.132-03	-.317-05	.000+00	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.379-01	.176-02	.138-06	.235-05	.95
					Method 2					
-.793+14	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.294+10	.161+12	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
-.265+05	-.215+05	-.115+08	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.235+01	.104+01	-.632-01	-.382+03	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.167+01	.156+01	.124+01	.557+00	.834-01	.000+00	.000+00	.000+00	.000+00	.000+00	
.152+01	.141+01	.116+01	.733+00	.174+00	-.643-03	.000+00	.000+00	.000+00	.000+00	.55
.144+01	.133+01	.110+01	.757+00	.344+00	.350-01	-.200-03	.000+00	.000+00	.000+00	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.368-02	.230-04	.000+00	.000+00	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.242-01	.111-03	-.295-03	.000+00	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.379-01	.176-02	.505-06	.929-05	.95
					Method 3					
.555+01	-.440+00	-.437+00	.665+00	-.617+00	.513+00	-.461+00	.423+00	-.327-02	.312-01	
.388+01	.127+01	-.178+00	.331-01	.673-02	-.128-01	.694-02	.463-02	-.123-01	-.731-02	.15
.249+01	.210+01	.523+00	-.202+00	.135+00	-.100+00	.880-01	-.830-01	.552-02	-.295-02	
.191+01	.188+01	.112+01	.127+00	-.542-01	.368-01	-.261-01	.159-01	.640-02	-.507-02	.35
.167+01	.155+01	.125+01	.533+00	-.225-01	.197-01	-.185-01	.196-01	-.390-02	-.886-03	
.153+01	.140+01	.117+01	.725+00	.186+00	-.187-01	.108-01	-.631-02	-.143-02	-.125-02	.55
.144+01	.133+01	.110+01	.759+00	.342+00	.392-01	-.480-03	-.215-02	.122-02	.500-03	
.139+01	.128+01	.106+01	.752+00	.406+00	.118+00	.314-02	.109-02	-.206-05	.817-04	.75
.136+01	.125+01	.104+01	.745+00	.427+00	.162+00	.243-01	.108-03	-.923-04	-.422-04	
.134+01	.124+01	.103+01	.741+00	.434+00	.179+00	.380-01	.175-02	-.620-05	-.588-05	.95

TABLE VII

Electron densities in aluminum from a 1.0 MeV plane isotropic source using N=10.

x=.15		x=.35		x=.55		x=.75		x=.95		s
Method 1										
.174+01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.366+01	-.591+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.237+01	.223+01	-.174+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.182+01	.175+01	.140+01	.977-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.155+01	.149+01	.130+01	.699+00	.480-01	.000+00	.000+00	.000+00	.000+00	.000+00	
.139+01	.133+01	.118+01	.844+00	.271+00	.341-01	.000+00	.000+00	.000+00	.000+00	.55
.131+01	.124+01	.109+01	.841+00	.453+00	.744-01	.103-01	.000+00	.000+00	.000+00	
.125+01	.118+01	.104+01	.813+00	.508+00	.190+00	.848-02	-.647-02	.000+00	.000+00	.75
.123+01	.116+01	.101+01	.792+00	.518+00	.244+00	.526-01	-.226-02	-.105-01	.000+00	
.121+01	.114+01	.100+01	.784+00	.518+00	.260+00	.770-01	.332-02	.136-02	-.137-02	.95
Method 2										
.174+01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.366+01	-.591+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.237+01	.223+01	-.174+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.182+01	.175+01	.140+01	.977-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.155+01	.149+01	.130+01	.699+00	.480-01	.000+00	.000+00	.000+00	.000+00	.000+00	
.139+01	.133+01	.118+01	.844+00	.271+00	.341-01	.000+00	.000+00	.000+00	.000+00	.55
.131+01	.124+01	.109+01	.841+00	.453+00	.744-01	.103-01	.000+00	.000+00	.000+00	
.125+01	.118+01	.104+01	.813+00	.508+00	.190+00	.848-02	-.647-02	.000+00	.000+00	.75
.123+01	.116+01	.101+01	.792+00	.518+00	.244+00	.526-01	-.226-02	-.105-01	.000+00	
.121+01	.114+01	.100+01	.784+00	.518+00	.260+00	.770-01	.332-02	.136-02	-.137-02	.95
Method 3										
.341+01	.209+01	.331+00	-.729+00	-.570+00	.238+00	.512+00	-.161+00	-.374+00	.541+00	
.304+01	.198+01	.536+00	-.414+00	-.431+00	.113+00	.351+00	-.847-01	-.259+00	.361+00	.15
.251+01	.181+01	.803+00	.204-01	-.216+00	-.266-01	.141+00	-.415-03	-.111+00	.140+00	
.202+01	.163+01	.101+01	.405+00	.150-01	-.929-01	-.215-01	.421-01	.165-02	-.172-01	.35
.165+01	.147+01	.112+01	.660+00	.216+00	-.660-01	-.952-01	.376-01	.536-01	-.786-01	
.142+01	.134+01	.113+01	.783+00	.363+00	.219-01	-.901-01	.106-01	.558-01	-.711-01	.55
.130+01	.125+01	.109+01	.816+00	.454+00	.122+00	-.411-01	-.112-01	.350-01	-.395-01	
.125+01	.119+01	.105+01	.808+00	.499+00	.200+00	.157-01	-.152-01	.143-01	-.145-01	.75
.122+01	.116+01	.101+01	.792+00	.515+00	.244+00	.573-01	-.637-02	.407-02	-.445-02	
.121+01	.114+01	.100+01	.783+00	.519+00	.261+00	.768-01	.266-02	.216-02	-.294-02	.95

TABLE VIII

Electron densities in aluminum from a 1.0 MeV plane isotropic source using N=20.

Method 1

x=.15		x=.35		x=.55		x=.75		x=.95		s
.668+05	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.364+01	-.113-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.235+01	.223+01	.386-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.182+01	.175+01	.139+01	-.357-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.155+01	.149+01	.129+01	.705+00	-.619-02	.000+00	.000+00	.000+00	.000+00	.000+00	
.139+01	.133+01	.118+01	.845+00	.266+00	-.990-03	.000+00	.000+00	.000+00	.000+00	.55
.130+01	.124+01	.109+01	.838+00	.458+00	.679-01	.109-02	.000+00	.000+00	.000+00	
.125+01	.118+01	.104+01	.812+00	.509+00	.190+00	.979-02	.476-03	.000+00	.000+00	.75
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.513-01	.538-03	-.266-03	.000+00	
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.745-01	.572-02	.574-05	.782-04	.95

Method 2

x=.15		x=.35		x=.55		x=.75		x=.95		s
.618+08	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.364+01	-.113-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
.235+01	.223+01	.386-01	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.182+01	.175+01	.139+01	-.357-02	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
.155+01	.149+01	.129+01	.705+00	-.619-02	.000+00	.000+00	.000+00	.000+00	.000+00	
.139+01	.133+01	.118+01	.845+00	.266+00	-.990-03	.000+00	.000+00	.000+00	.000+00	.55
.130+01	.124+01	.109+01	.838+00	.458+00	.679-01	.109-02	.000+00	.000+00	.000+00	
.125+01	.118+01	.104+01	.812+00	.509+00	.190+00	.979-02	.476-03	.000+00	.000+00	.75
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.513-01	.538-03	-.266-03	.000+00	
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.745-01	.572-02	.574-05	.782-04	.95

Method 3

x=.15		x=.35		x=.55		x=.75		x=.95		s
.535+01	.171-01	-.952+00	.786+00	-.372+00	-.149-01	.223+00	-.256+00	.558-01	.517+00	
.381+01	.133+01	-.167+00	.942-02	.361-01	-.445-01	.412-01	-.352-01	.179-01	.419-01	.15
.242+01	.206+01	.632+00	-.234+00	.807-01	.150-01	-.585-01	.624-01	-.153-01	-.116+00	
.178+01	.187+01	.120+01	.199+00	-.750-01	.373-01	-.183-01	.114-01	-.103-01	-.179-02	.35
.154+01	.150+01	.130+01	.659+00	.112-01	-.201-01	.267-01	-.240-01	.667-02	.373-01	
.140+01	.131+01	.119+01	.837+00	.269+00	-.189-01	.851-02	-.549-02	.500-02	.146-02	.55
.130+01	.124+01	.109+01	.841+00	.450+00	.816-01	-.122-01	.742-02	-.219-02	-.806-02	
.125+01	.119+01	.104+01	.813+00	.510+00	.188+00	.115-01	.832-03	-.134-02	-.540-03	.75
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.520-01	-.419-03	.346-03	.748-03	
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.746-01	.562-02	.129-03	.181-03	.95

TABLE IX

Electron densities in aluminum from a 1.0 MeV plane isotropic source using N=34.

x=.15		x=.35		x=.55		x=.75		x=.95		s
				Method 1						
.668+05	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
-.172+04	.220+05	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
-.233+07	-.185+07	.162+08	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
-.410+06	.166+07	.145+07	.131+08	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.512+03	-.463+03	-.594+02	-.679+03	.506+04	.000+00	.000+00	.000+00	.000+00	.000+00	.55
.266+01	.262+01	.230+01	.472+00	.432+00	.970+01	.000+00	.000+00	.000+00	.000+00	
.131+01	.124+01	.109+01	.833+00	.456+00	.768-01	.514-01	.000+00	.000+00	.000+00	.75
.125+01	.118+01	.104+01	.812+00	.509+00	.190+00	.976-02	.551-03	.000+00	.000+00	
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.512-01	.543-03	.342-05	.000+00	.95
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.745-01	.574-02	.324-05	.204-05	
				Method 2						
-.147+21	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.15
-.149+12	.659+18	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.133+11	.189+13	-.126+18	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	.35
-.410+06	.459+07	.951+10	.520+13	.000+00	.000+00	.000+00	.000+00	.000+00	.000+00	
.512+03	-.499+03	-.542+05	-.303+08	-.371+10	.000+00	.000+00	.000+00	.000+00	.000+00	.55
.266+01	.261+01	-.463+01	-.301+03	.567+05	.730+07	.000+00	.000+00	.000+00	.000+00	
.131+01	.124+01	.109+01	.435+00	-.128+02	.335+03	.396+05	.000+00	.000+00	.000+00	.75
.125+01	.118+01	.104+01	.812+00	.490+00	-.100+01	-.434+02	-.104+04	.000+00	.000+00	
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.624-01	.452+00	.907+01	.000+00	.95
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.745-01	.608-02	.604-02	.127+00	
				Method 3						
.542+01	-.135+00	-.726+00	.814+00	-.620+00	.436+00	-.388+00	.453+00	-.187+00	-.104+00	
.382+01	.132+01	-.162+00	.160-01	.182-01	-.194-01	.100-01	.691-02	-.180-01	-.125-01	.15
.240+01	.210+01	.648+00	-.239+00	.138+00	-.873-01	.772-01	-.940-01	.429-01	.243-01	
.178+01	.187+01	.121+01	.195+00	-.774-01	.466-01	-.317-01	.217-01	.164-02	.265-02	.35
.155+01	.148+01	.131+01	.651+00	-.315-02	.129-01	-.155-01	.244-01	-.148-01	-.863-02	
.140+01	.132+01	.119+01	.839+00	.271+00	-.235-01	.145-01	-.101-01	.568-03	-.340-03	.55
.130+01	.124+01	.109+01	.841+00	.454+00	.750-01	-.418-02	-.192-02	.323-02	.195-02	
.125+01	.119+01	.104+01	.812+00	.510+00	.189+00	.103-01	.175-02	-.662-03	-.223-03	.75
.123+01	.116+01	.101+01	.793+00	.520+00	.243+00	.513-01	.320-03	-.156-03	-.130-03	
.121+01	.114+01	.998+00	.784+00	.521+00	.260+00	.745-01	.573-02	.167-04	-.773-06	.95

finite number of polynomials. This gives one incentive for using method 1 or 2. Alternatively, method 3 could be used to reconstruct the density of collided electrons which would be added to an analytic expression for the density of uncollided electrons. (The discontinuities at the wave front including the delta function at s=0 arise from uncollided electrons.) In order to cut out some of the nonphysical oscillation in the region $|x| > s$, s small, Figs. 2, 4 and 6 give $\phi_0(x,s)$ for $0.25 \leq s \leq .981$.

It is reassuring that method 3 is able to locate the wave front. This is more apparent from Figs. 5 and 6 then from Figs. 1 to 4. Only uncollided electrons appear at the wave front. From Eq. (11), $\sigma(s) \to \infty$ as $s \to 1$. Thus the wave front disappears in Figs. 1 to 4 as s increases. To verify that method 3 correctly locates the wave front the mean-free-path of the electrons corresponding to Figs. 5 and 6 was set equal to 1.0 (range units).

V. CONCLUSIONS

A closed form solution for the angular-spatial moments of the Spencer Lewis Equation has been found. The validity of the solution was checked in two ways. First, the $\phi_{\ell n}(s)$ were integrated to yield the $\phi^{p}_{\ell n}$. These agree to at least five significant figures with those obtained directly from Spencer's recursion relation, Eq. (13), and they are in good agreement with

Fig. 1 Electron density for $s \geq .025$ due to isotropic plane source of 0.1 MeV electrons located at $s = x = 0$ in aluminum.

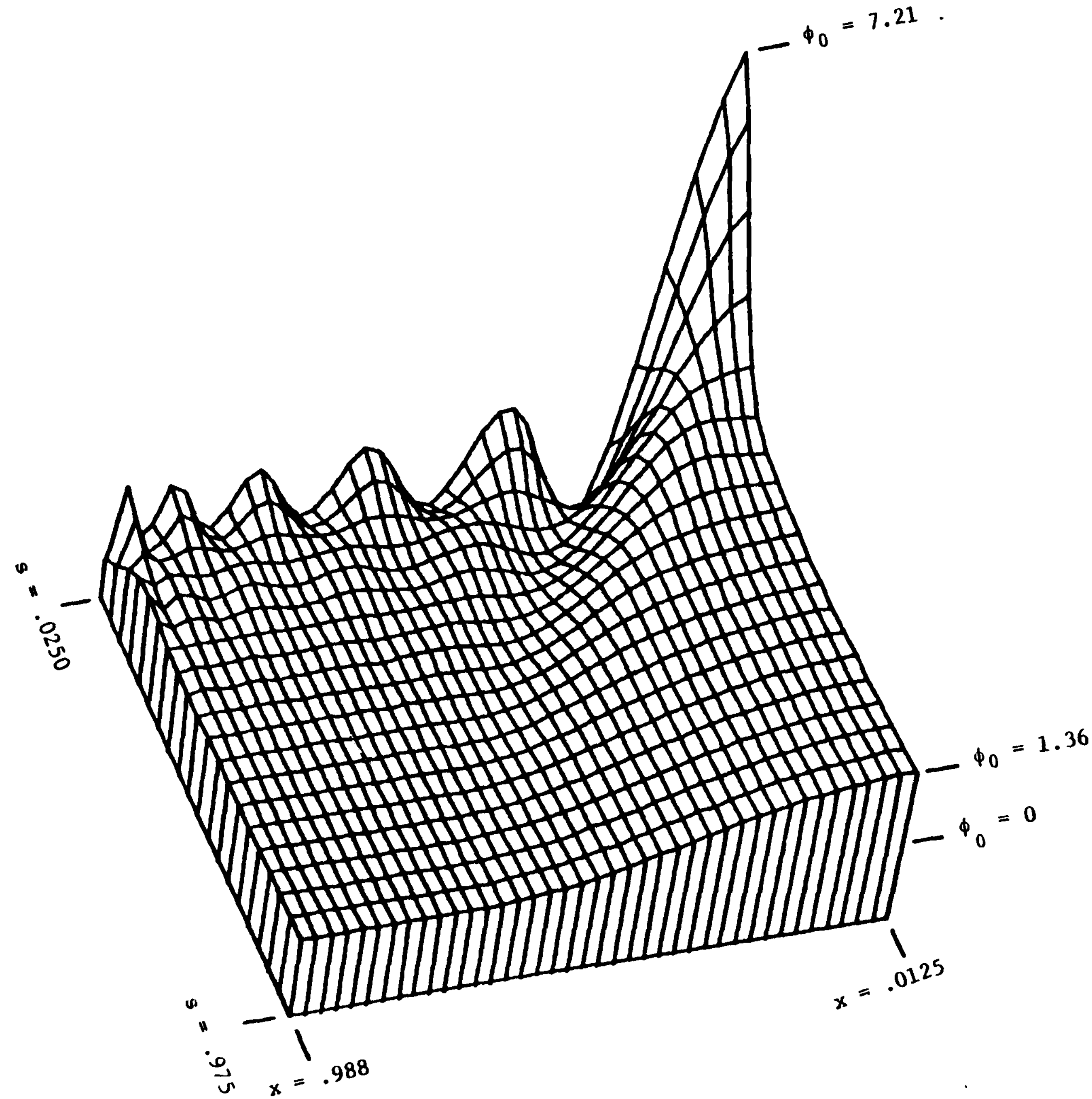


Fig. 2 Electron density for s ≥ .25 due to an isotropic plane source of 0.1 MeV electrons located at s = x = 0 in aluminum.

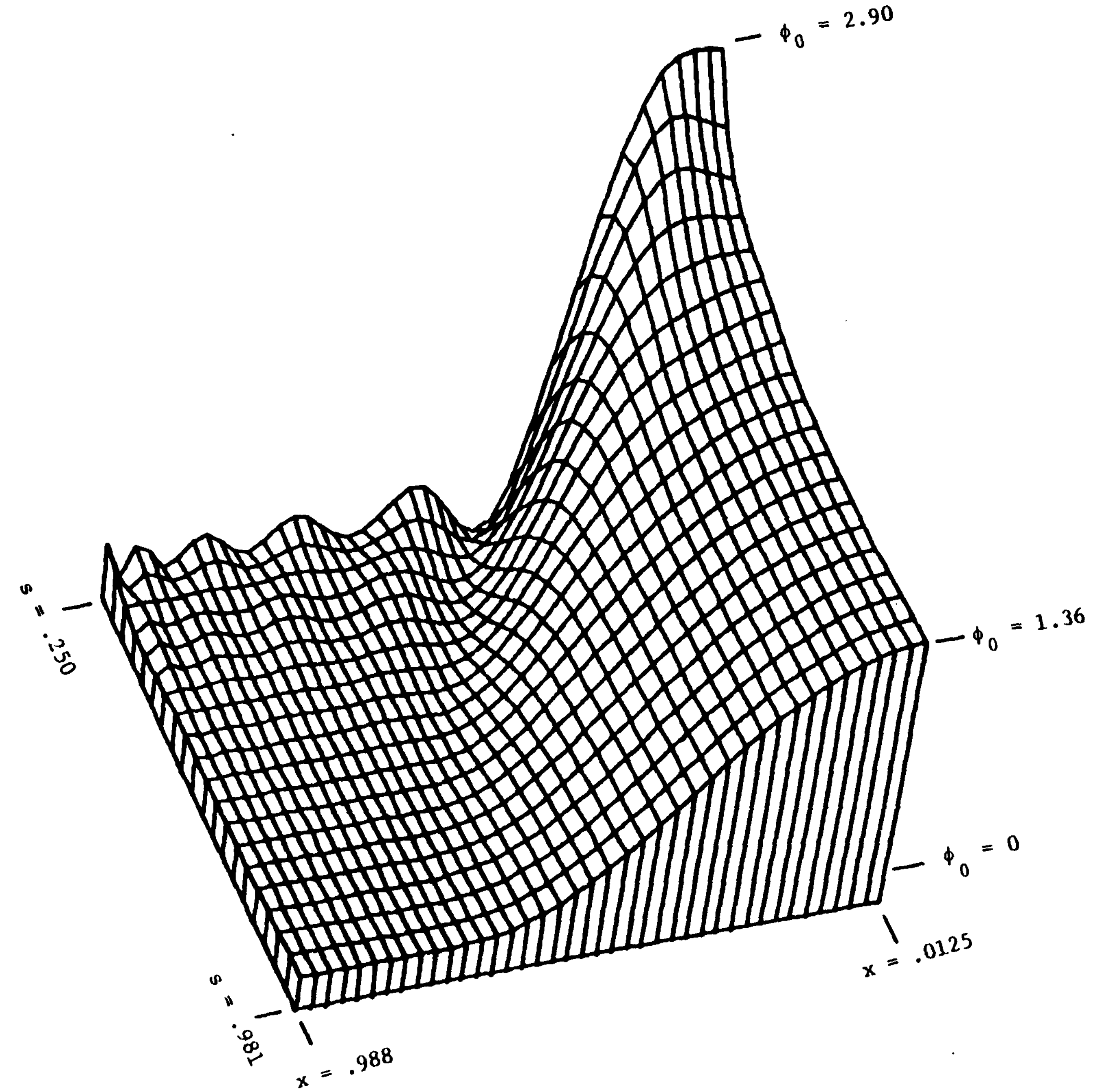


Fig. 3 Electron density for $s \geq .025$ due to an isotropic plane source of 1.0 MeV electrons located at $s = x = 0$ in aluminum.

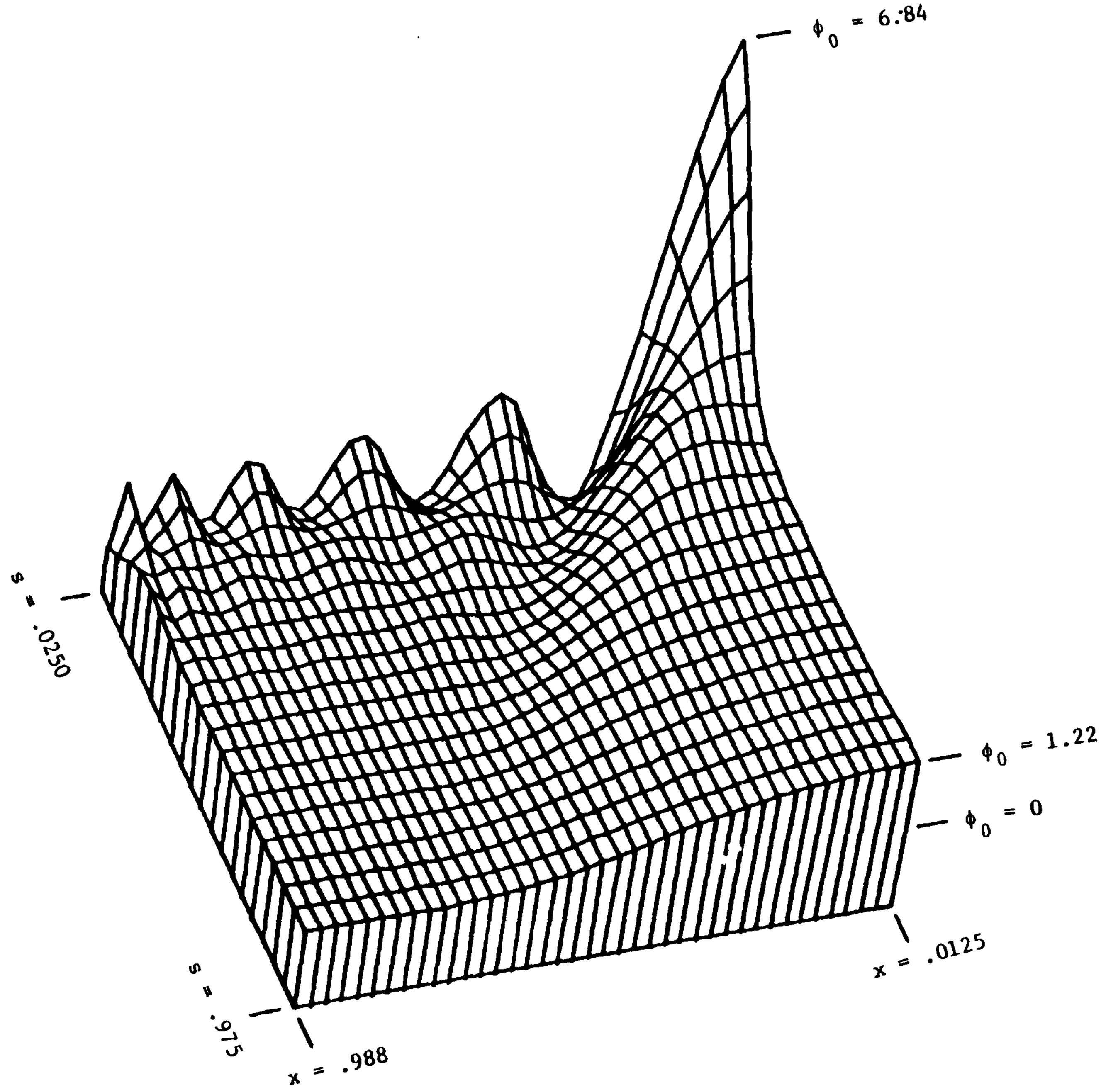


Fig. 4 Electron density for $s \geq .25$ due to an isotropic plane source of 1.0 MeV electrons located at $s = x = 0$ in aluminum.

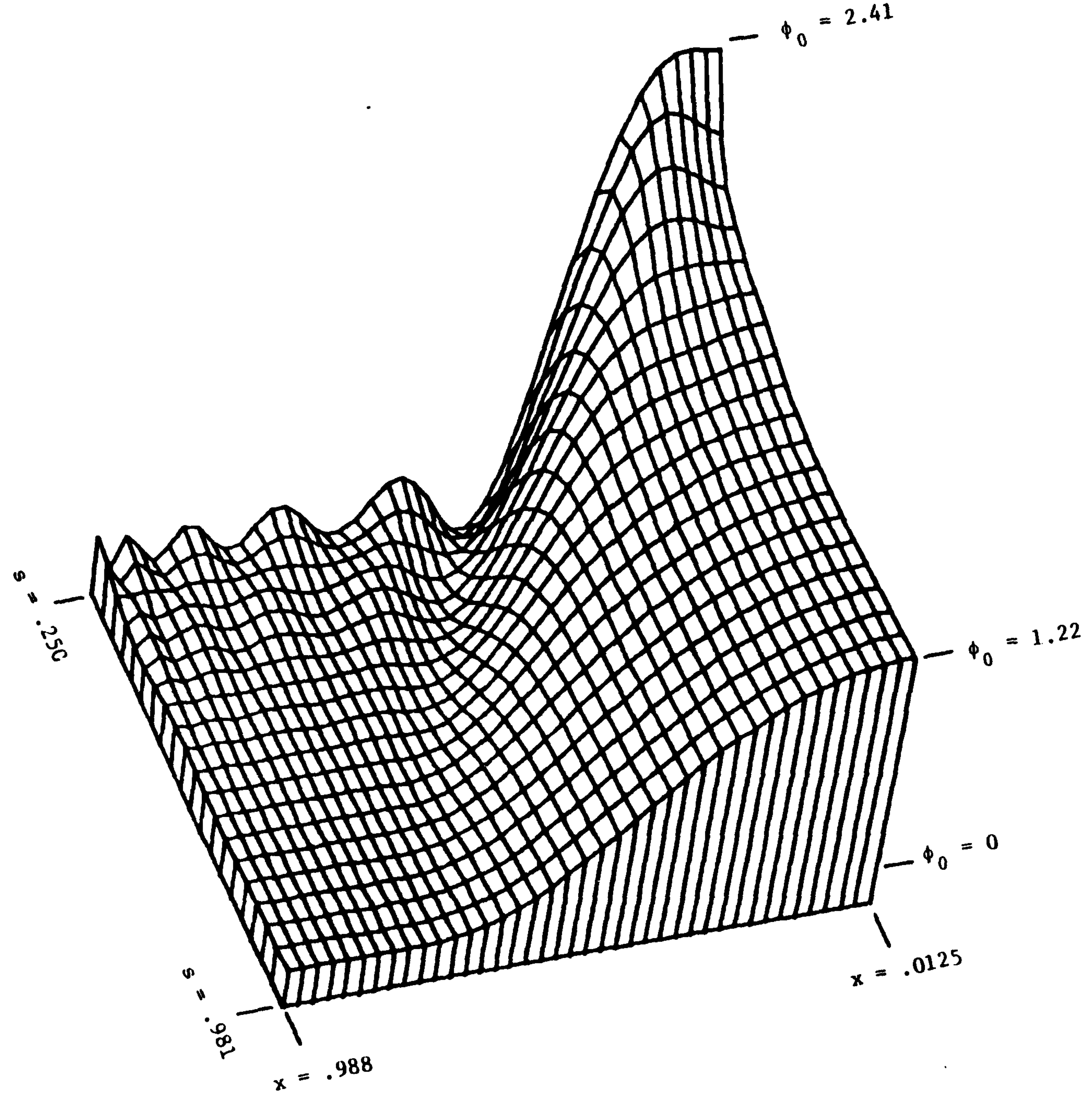


Fig. 5 Electron density for $s \geq .025$ due to an isotropic plane source of electrons with a range of 1.0 mean-free-path in a fictitious material with f_ℓ's equal to those for 1.0 MeV electrons in aluminum.

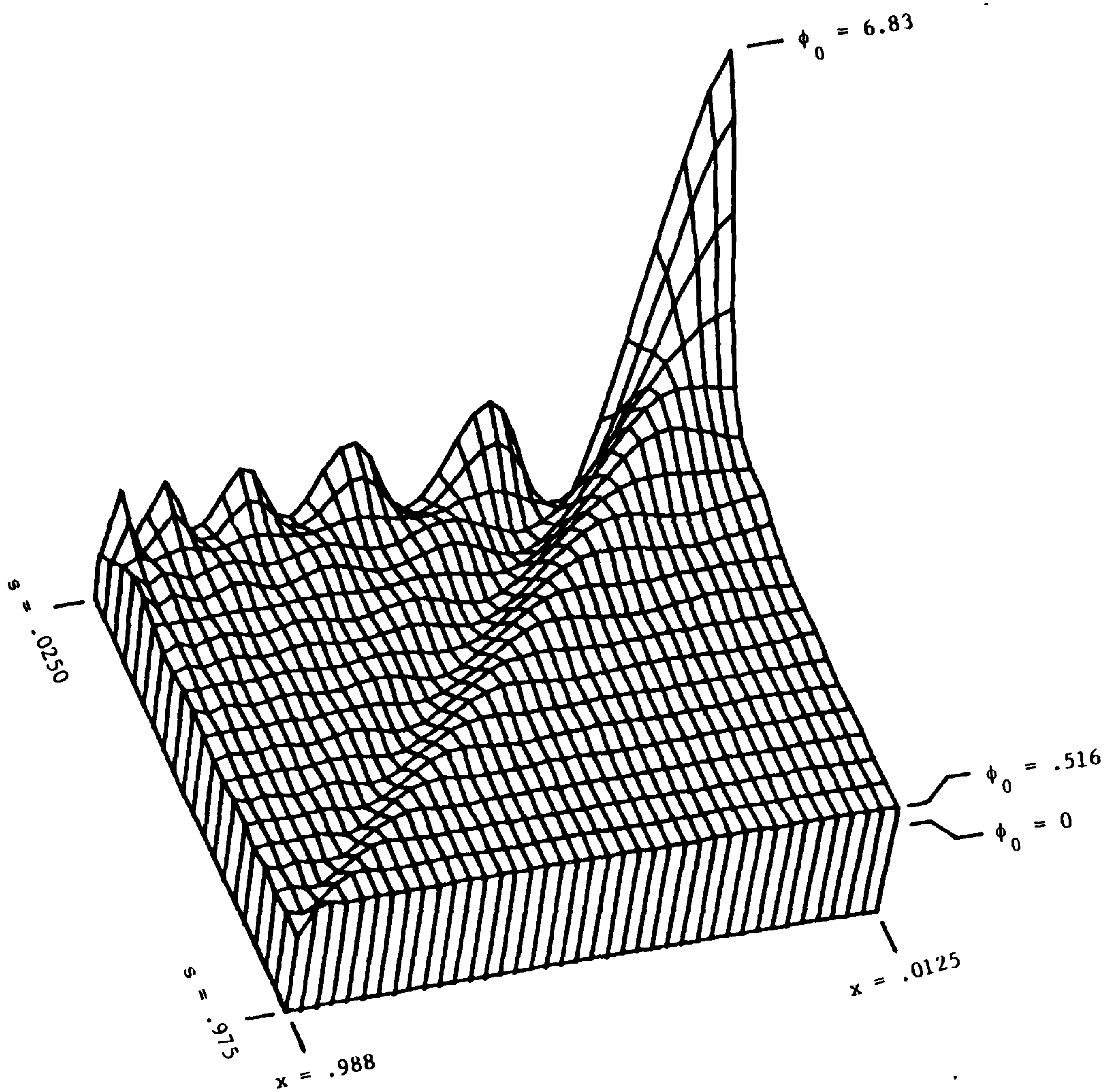
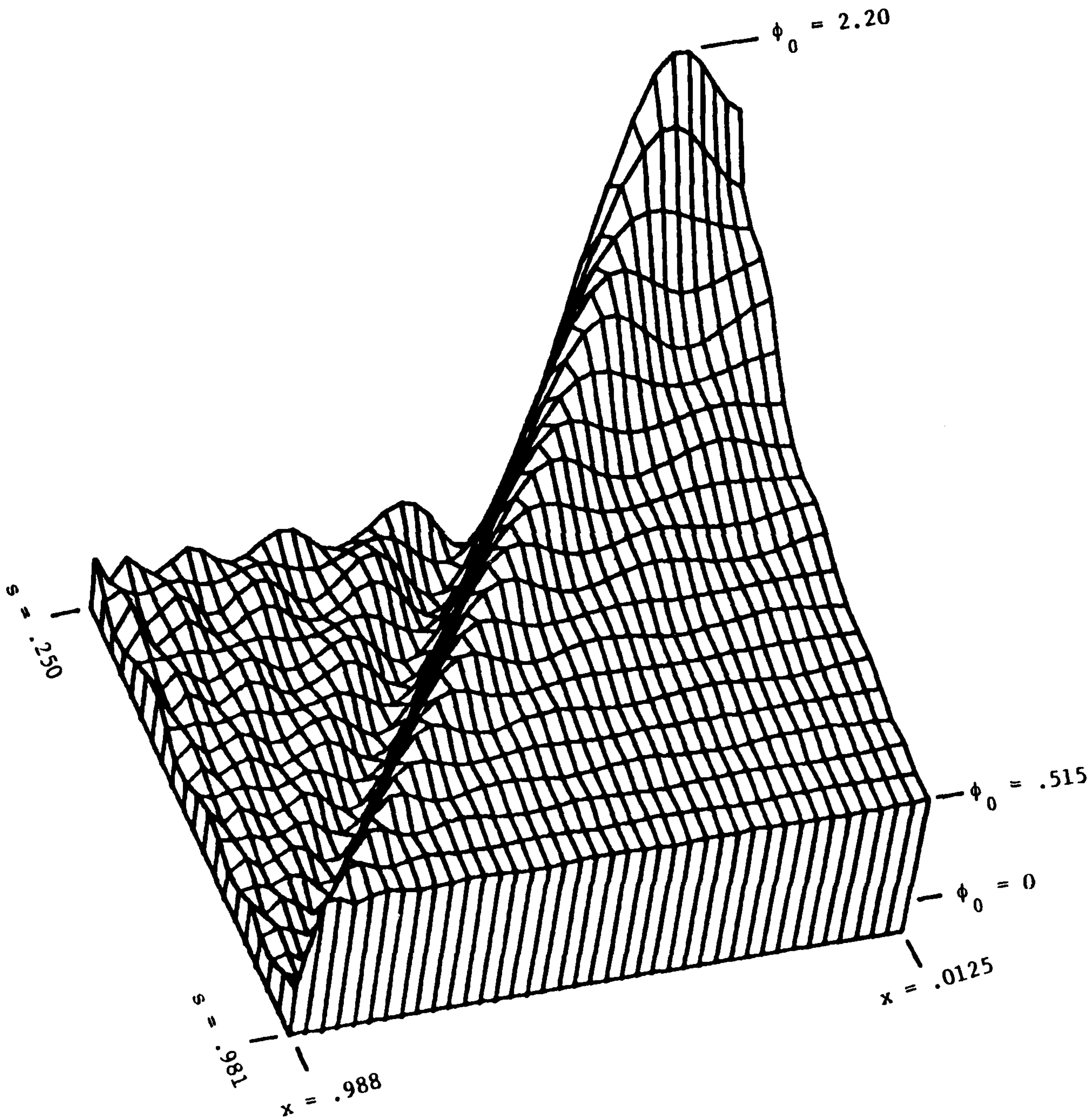


Fig. 6 Electron density for $s \geq .25$ due to an isotropic plane source of electrons with a range of 1.0 mean-free-path in a fictitious material with f_ℓ's equal to those for 1.0 MeV electrons in aluminum.



Spencer's published results[10] for large α values (Eq. (50) corresponds to the case $\alpha = \infty$). Second, the electron density, $\phi_0(x,s)$, was reconstructed from the $\phi_{0n}(s)$. Although no independent calculations are currently available for comparison, the results look reasonable and the location of the wave front is correctly predicted.

Since the solution for the $\phi_{\ell n}(s)$ are valid only for the approximate s dependence of the cross sections specified by Eq. (11), and for homogeneous infinite media with plane sources of electrons, their use for generating solutions to practical engineering problems is limited. On the other hand, they enable analytic benchmark solutions for $\phi(x,s)$ and in theory for $\phi(x,s,\mu)$. This is expected to be the main benefit of this project.

Several interesting observations are possible as a result of the analytic expressions for the $\phi_{\ell n}(s)$. The exponents in Eq. (23) are all positive except for the case $\ell=i=j=0$ for which the exponent is zero. Therefore,

$$\lim_{s \to 1} \phi_{\ell n}(s) = \begin{cases} A_{0n00}, & \ell=0, \\ 0, & \text{otherwise}, \end{cases} \tag{51}$$

implying [see Eqs. (4) and (10)] that the electron distribution becomes isotropic as $s \to 1$. The physical reason for this is that the differential scattering cross section blows up as $s \to 1$ [see Eq. (11)].

From Eqs. (17), (23) and the last line of Eq. (27),

$$\lim_{s \to 0^+} \phi_{\ell n}(s) = \begin{cases} q_\ell, & n=0 \\ 0, & n>0 \end{cases} \tag{52}$$

which is in agreement with Eq. (21).

The total number of electrons in the medium for a given s value is given by

$$\int_{-1}^{1} \phi_0(x,s)\, dx = \phi_{00}(s) = q_0\, \theta(s) \tag{53}$$

where use has been made of Eqs. (10) and (17).

Therefore, $\phi(x,s)$ does not tend to zero at the electron range, that is, as $s \to 1$. In spite of the infinite scattering cross section, there is no mechanism in the model for eliminating electrons, and $\phi_{00}(s)$ remains constant. (For $s > 1$ the scattering cross section, Eq. (11), goes negative and Eq. (22) predicts complex solutions.)

As a final point, Figs. 1 and 2 for 0.1 MeV electrons are practically identical to Figs. 3 and 4 for 1.0 MeV electrons. Some of the similarity is due to scaling and the fact that distances are given in electron range units. Further studies are under way to determine if any additional explanations can be found.

IV. Suggested Follow-on Work

From the results obtained so far, it appears that the following additional work should be carried out.

A. Modify the computer code such that the distributions of uncollided electrons is determined analytically, while moments reconstruction is used only for the density of collided electrons. This should speed convergence for Methods 1, 2 and 3 of Section II.C and eliminate the oscillations in the region $|x|>s$ obtained with Method 3.

B. Compare the electron densities determined with the modified code of Section IV.A with those obtained by an independent method such as the streaming ray numerical technique.

C. Equations (23) to (29) constitute an exact solution of the Spencer-Lewis angular-spatial-moments equations and should be reported in a scientific journal. This would require writing a suitable introduction to Sections II to V which are already in journal format. In addition, some modifications based on the results of VI.A and VI.B would also be needed.

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